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Hölder Estimates on CR Manifolds with a Diagonalizable Levi Form

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0. INTRODUCTION

The purpose of this paper is to prove the following result. Let $n \geqslant 3$.

MAIN THEOREM. *Let $\mathscr{M}$ be a compact pseudoconvex $(2n-1)$-dimensional, CR manifold. Let $P_0 \in \mathscr{M}$ be a point of type m and suppose that there is a neighborhood of P_0 on which the Levi form is diagonalizable. Suppose further that the range of $\bar\partial_b$ is closed in L_2, then*

(a) *For all $\varepsilon > 0$ the operator $\square_b^{-1}$ maps* $\mathrm{Lip}(s, P_0)$ *into* $\mathrm{Lip}(s + 2/m - \varepsilon, P_0)$. *Here* $\mathrm{Lip}(s, P_0)$ *denotes the space of square-integrable forms (of degree (p, q) with $1 \leqslant q \leqslant n-2$) which are Lipschitz of order s in a neighborhood of P_0.*

(b) *For all $\varepsilon > 0$ the operators $\bar\partial_b \square_b^{-1}$, $\bar\partial_b^* \square_b^{-1}$, $\square_b^{-1} \bar\partial_b$, and $\square_b^{-1} \bar\partial_b^*$ map* $\mathrm{Lip}(s, P_0)$ *into* $\mathrm{Lip}(s + 1/m - \varepsilon, P_0)$.

(c) *For all $\varepsilon > 0$ the operators $\bar\partial_b \bar\partial_b^* \square_b^{-1}$, $\bar\partial_b^* \bar\partial_b \square_b^{-1}$, $\square_b^{-1} \bar\partial_b \bar\partial_b^*$, $\square_b^{-1} \bar\partial_b^* \bar\partial_b$, $\bar\partial_b \square_b^{-1} \bar\partial_b^*$, and $\bar\partial_b^* \square_b^{-1} \bar\partial_b$ map* $\mathrm{Lip}(s, P_0)$ *into* $\mathrm{Lip}(s - \varepsilon, P_0)$. *Here again we assume $\square_b^{-1}$ acts on forms of type (p, q) with $1 \leqslant q \leqslant n-2$.*

(d) *If $\bar\partial_b u = \alpha$, if u is orthogonal to the nullspace of $\bar\partial_b$, and if α is a $(0, 1)$-form in* $\mathrm{Lip}(s, P_0)$ *then $u \in$* $\mathrm{Lip}(s + 1/m - \varepsilon, P_0)$ *for all $\varepsilon > 0$.*

(e) *If f is a square-integrable function on $\mathscr{M}$ with $f \in$* $\mathrm{Lip}(s, P_0)$ *then $S_b(f) \in$* $\mathrm{Lip}(s - \varepsilon, P_0)$ *for all $\varepsilon > 0$ where S_b is the orthogonal projection on square-integrable CR functions.*

The following is a consequence of the main theorem.

COROLLARY. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with a smooth boundary $\mathscr{M}$. Let $P_0 \in \mathscr{M}$ be a point of type m and suppose that there is a neighborhood of P_0, in $\mathscr{M}$ on which the Levi form is diagonalizable.*

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Given a form α on Ω, we will denote its restriction to the boundary $\mathcal{M}$ by α_b. Then we have

(a) *If $\bar{\partial} u = \alpha$ with α a $(0, 1)$-form and $\alpha \in \mathrm{Lip}(s, P_0)$, α_b, $u_b \in L_2(\mathcal{M})$ and u_b orthogonal to the nullspace of $\bar{\partial}_b$, then $u \in \mathrm{Lip}(s + 1/m - \varepsilon, P_0)$ for all $\varepsilon > 0$. Here $\alpha \in \mathrm{Lip}(s, P_0)$ means that there exists a neighborhood U of P_0 in $\mathbb{C}^n$ such that α is Lipschitz of order s on $U \cap \bar{\Omega}$.*

(b) *If f is a function on $\bar{\Omega}$ whose restriction to $\mathcal{M}$ is square-integrable, then we denote by Sf the Szego projection of f (i.e., Sf is the holomorphic function on Ω whose boundary values are $S_b f$). Suppose that $f \in \mathrm{Lip}(s, P_0)$ then $S(f) \in \mathrm{Lip}(s - \varepsilon, P_0)$ for all $\varepsilon > 0$.*

Observe that the condition diagonalizable Levi form is automatically satisfied if $\dim \mathcal{M} = 3$ and more generally if all but at most one of the eigenvalues of the Levi form are positive. Results proving analogous regularity theorems for Sobolev norms were obtained in [K1] in the strongly pseudoconvex case and in [K2] in general.

For strongly pseudoconvex CR manifolds the above theorem (in fact with $\varepsilon = 0$) is proved in G. Folland and E. M. Stein [FoS] and in P. Greiner and E. M. Stein [GS]. In the strongly pseudoconvex case the above Corollary was proved by Phong in [P]. When $\mathcal{M}$ is three-dimensional the above theorem holds with $\varepsilon = 0$ and for $(0, 1)$-forms, see [Ch1, Ch2, FK1, FK2, CNS, NRSW]. If $n \geqslant 3$ and if the Levi form has exactly one eigenvalue that is not strictly positive, results closely related to the theorem were proved by M. Machedon in [M1, M2]. The operator $\bar{\partial}_b$ on real ellipsoids has been analyzed by Shaw in [S2].[1]

In this introduction we will show how the Corollary follows from the Main Theorem and we will give a sketch of the strategy that we use in the proof of the Main Theorem. First, however, we will define the terms used in the statements of the above results.

0.1. Definition. Let $\mathcal{M}$ be a $(2n - 1)$-dimensional manifold. Then a *CR structure* on $\mathcal{M}$ is given by a subbundle $T^{1,0}(\mathcal{M})$ of the complex tangent bundle $\mathbb{C}T(\mathcal{M})$ satisfying the following properties:

(a) $T^{1,0}(\mathcal{M}) \cap \overline{T^{1,0}(\mathcal{M})} = \{0\}$.

(b) The fiber dimension of $T^{1,0}(\mathcal{M})$ is $n - 1$.

(c) If L and L' are local vector fields with values in $T^{1,0}(\mathcal{M})$, then the commutator $[L, L'] = LL' - L'L$ also has values in $T^{1,0}(\mathcal{M})$.

A manifold $\mathcal{M}$ with a fixed CR structure is called a *CR manifold*.

[1] *Note added in proof.* Recently McNeal proved that, on a class of pseudo-convex domains, d'Angelo's finite type condition is necessary for a gain of Hölder regularity of $\bar{\partial}$ (see [Mc]).

0.2. DEFINITION. Let $L_1, \ldots, L_{n-1}$ be C^∞ vector fields on an open set $U \subset \mathcal{M}$ which are a local basis of sections of $T^{1,0}(\mathcal{M})$ on U. Let T be a real vector field on U such that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}, T$ is a basis of the complex vector fields. The vector field $[L_i, \bar{L}_j]$ in terms of this basis is given by

$$[L_i, \bar{L}_j] = -c_{ij}\sqrt{-1}\, T + \sum a_{ij}^k L_k + \sum b_{ij}^k \bar{L}_k. \tag{1}$$

The hermitian form (c_{ij}) is called the *Levi-form.* $\mathcal{M}$ is called *pseudoconvex* if each point of $\mathcal{M}$ has a neighborhood on which the vector field T can be chosen so that $(c_{ij}) \geqslant 0$. The Levi form is said to be *diagonalizable* on U if the local basis $L_1, \ldots, L_{n-1}$ can be chosen so that

$$c_{ij} = \delta_{ij}\lambda_i \tag{2}$$

on U.

0.3. *Remark.* If $\mathcal{M}$ is a hypersurface in $\mathbb{C}^n$ then it has the CR structure induced by $\mathbb{C}^n$ where $T^{1,0}(\mathcal{M}) = T^{1,0}(\mathbb{C}^n) \cap \mathbb{C}T(\mathcal{M})$; that is, the fiber $T_P^{1,0}(\mathcal{M})$ consists of vectors of the form $\sum a_i(\partial/\partial z_i)$ which are tangent to $\mathcal{M}$ at P.

0.4. *Remark.* Let $P_0 \in \mathcal{M} \subset \mathbb{C}^n$, r a local defining of $\mathcal{M}$ near P_0, that is, $r=0$ on $\mathcal{M}$ and $dr \neq 0$, and let $z_1, \ldots, z_n$ be coordinates with origin at P_0 such that $r_{z_n}(0) \neq 0$. Define L_j and T by

$$\begin{aligned} L_j &= \frac{\partial}{\partial z_j} - \frac{r_{z_j}}{r_{z_n}} \frac{\partial}{\partial z_n}, \qquad j = 1, \ldots, n-1 \\ T &= -\sqrt{-1}\left(r_{\bar{z}_n} \frac{\partial}{\partial z_n} - r_{z_n} \frac{\partial}{\partial \bar{z}_n} \right). \end{aligned} \tag{3}$$

Then in (1) we have $a_{ij}^k = b_{ij}^k = 0$. If the Levi-form is diagonalizable then in general a basis which diagonalizes the Levi form does not have $a_{ij}^k = b_{ij}^k = 0$.

A point P_0 in a pseudoconvex hypersurface $\mathcal{M} \subset \mathbb{C}^n$ is of finite type if the orders of contact of all complex analytic curves through P_0 with $\mathcal{M}$ are bounded. Catlin in [C1] proved that finite type is a necessary condition for the subellipticity of the $\bar{\partial}$-Neumann problem. D'Angelo in [D'A] discovered the basic geometric properties of finite type and they were used by Catlin in [C2] to prove that finite type is also sufficient for subellipticity. R. Diaz in [D] shows that finite type is necessary for subellipticity of $\square_b$. In [K4] another finiteness condition is introduced which is shown in [K2] to be sufficient for subellipticity of $\square_b$. This finiteness condition is equivalent to finite type in the case of real analytic $\mathcal{M} \subset \mathbb{C}^n$ and in case the Levi form is diagonalizable see [K3]. When the Levi form is diagonalizable finite type is also equivalent to the condition (see Post [Po]) introduced in [K4] and which we give in the following definition. Our method of proof is based on the study of certain real, second-order pseudodifferential operators (similar to those stated in [FK1]), these operators are closely related to the second-order operators studied by Hörmander in [H], Rothschild and Stein in [RS], and Fefferman and Sanchez-Calle in [FS].

0.5. DEFINITION. Suppose that the Levi form is diagonalizable in a neighborhood of $P_0 \in \mathcal{M}$ and that $L_1, ..., L_{n-1}$ is the basis for which (2) holds then we say that P_0 is of *finite type* if for each $i = 1, ..., n-1$ the Lie algebra generated by L_i and $\bar{L}_i$ contains T modulo $L_1, ..., L_{n-1}$, $\bar{L}_1, ..., \bar{L}_{n-1}$. This is also equivalent to the condition that for each i there exists a monomial P_i in L_i, $\bar{L}_i$ such that

$$p_i(L_i, \bar{L}_i)\lambda_i|_{P_0} \neq 0. \tag{4}$$

The type of L_i denoted by m_i is given by

$$m_i = \deg p_i + 2, \tag{5}$$

where p_i is the polynomial of the smallest degree for which (4) holds. Equivalently, m_i is the least number of brackets needed to express $T \bmod (L_1, ..., \bar{L}_{n-1})$ in the Lie algebra generated by $(L_i, \bar{L}_i)$. Finally, we define m, the type of P_0, by

$$m = \max\{m_i\}. \tag{6}$$

0.6. *Remark.* The above definition is sharp for the case of $(p, 1)$-forms, for (p, q)-forms it suffices to use the following weaker condition. $P_0 \in \mathcal{M}$ is of *finite q-type* if every subset $L_{i_1}, ..., L_{i_k}$ of $L_1, ..., L_{n-1}$ with $k = \min(q, n-q-1)$ contains at least one element of finite type in the sense of (4). Set $m_{i_1 i_2 \cdots i_k} = \min\{m_{i_j}\}$ and $m^{(q)} = \max\{m_{i_1 \cdots i_k}\}$. The $m_{i_1 \cdots i_k}$ is the q-type at P_0. Equivalently, $m^{(q)}$ is the least number of brackets needed to express $T \bmod (L_1, ..., \bar{L}_{n-1})$ in the Lie algebra generated by $(L_{i_1}, \bar{L}_{i_1}, ..., L_{i_k}, \bar{L}_{i_k})$.

0.7. *Remark.* Pseudoconvexity implies that the numbers m_i are even.

0.8. DEFINITION. Setting $T^{0,1}(\mathcal{M}) = \overline{T^{1,0}(\mathcal{M})}$, we denote by $\mathcal{B}(\mathcal{M})$ the bundle of differential forms on $T^{1,0}(\mathcal{M}) \oplus T^{0,1}(\mathcal{M})$. This bundle can be expressed as a direct sum $\mathcal{B}(\mathcal{M}) = \bigoplus \mathcal{B}^{p,q}(\mathcal{M})$. Associated with $\mathcal{B}^{p,q}(\mathcal{M})$ is the natural exterior differential operator denoted by $\bar{\partial}_b$ in terms of the local basis $L_1, ..., L_{n-1}$ of $T^{1,0}(\mathcal{M})$ on U. Let $\omega^1, ..., \omega^{n-1}$ be the local basis of $\mathcal{B}^{1,0}(\mathcal{M})$ on U. Thus if φ is a (p, q)-form on U we have

$$\varphi = \sum_{I,J} \varphi_{IJ} \omega^I \wedge \bar{\omega}^J, \tag{7}$$

where I and J run over strictly increasing p-tuples and q-tuples, respectively, of integers in $\{1, ..., n-1\}$, $\varphi_{IJ} \in C^\infty(U)$,

$$\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \qquad \text{with} \quad I = (i_1, ..., i_p),$$

and

$$\bar{\omega}^J = \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} \qquad \text{with} \quad J = (j_1, ..., j_q).$$

Then

$$\bar{\partial}_b \varphi = \sum \bar{\partial}_b(\varphi_{IJ}) \wedge \omega^K \wedge \bar{\omega}^J + (-1)^p \sum \varphi_{IJ} \omega^I \wedge \bar{\partial}_b \bar{\omega}^J, \tag{8}$$

where

$$\bar{\partial}_b(\varphi_{IJ}) = \sum \bar{L}_k(\varphi_{IJ}) \bar{\omega}^k,$$

$$\bar{\partial}_b \bar{\omega}^J = \sum_k (-1)^{k+1} \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\partial}_b \bar{\omega}^{j_k} \wedge \cdots \wedge \bar{\omega}^{j_q},$$

and

$$\bar{\partial}_b \bar{\omega}^j = \sum \bar{a}^j_{rs} \bar{\omega}^r \wedge \bar{\omega}^s.$$

Here the coefficients a^j_{rs} are given by

$$[L_r, L_s] = \sum_j a^j_{rs} L_j.$$

0.9. Definition. Fix a hermitian metric on $\mathbb{C}T(\mathcal{M})$ such that the L_i's form an orthonormal basis and such that $T^{1,0}(\mathcal{M})$ is orthogonal to $T^{0,1}(\mathcal{M})$. This induces a metric on $\mathcal{B}(\mathcal{M})$ under which the $\mathcal{B}^{p,q}(\mathcal{M})$ are orthogonal. We then define an L_2 inner product on $\mathcal{B}^{p,q}(\mathcal{M})$. We denote $\bar{\partial}_b^*$: $\mathcal{B}^{p,q}(\mathcal{M}) \to \mathcal{B}^{p,q-1}(\mathcal{M})$ the L_2-adjoint of $\bar{\partial}_b$. Further, we define the Laplacian

$$\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b. \tag{9}$$

In [FK1] we defined the spaces $LIP(\alpha)$ for each $\alpha \in (-\infty, \infty)$. When $\alpha \geqslant 0$ these spaces are contained in the usual Lipschitz spaces and when $\alpha > 0$ and not an integer then these spaces equal the usual Lipschitz spaces. Here we will denote these spaces by $\mathrm{Lip}(\alpha)$ and we will recall the definition given in [FK1].

0.10. DEFINITION. Let $\psi \in C_0^\infty(\{\xi \in \mathbb{R}^{2n-1} \mid 0 < a < |\xi| < b\})$. For $u \in \bigcup_{s_0} H^{-s_0}(\mathbb{R}^{2n-1})$, $\delta > 0$, $M > 0$ we define $\Gamma_\delta u$ by $\widehat{\Gamma_\delta u(\xi)} = \psi((\delta/M)\xi)\,\hat{u}(\xi)$. For $-\infty < \alpha < \infty$ define $\mathrm{Lip}(\alpha)$ to be those $u \in \bigcup_s H^{-s}(\mathbb{R}^{2n-1})$ such that

$$\|\Gamma_\delta u\|_{L^\infty} \leqslant C\delta^\alpha \tag{10}$$

for all small $\delta > 0$. If $P_0 \in \mathcal{M}$ then a function $u \in \mathrm{Lip}(\alpha, P_0)$ if there exists $\zeta \in C_0^\infty(U)$, where U is a coordinate neighborhood, containing P_0, such that $\zeta = 1$ in a neighborhood of P_0 and such that $\zeta u \in \mathrm{Lip}(\alpha)$. A differential form is said to be in $\mathrm{Lip}(\alpha, P_0)$ if its coefficients are in $\mathrm{Lip}(\alpha, P_0)$.

Next we will show that the Corollary is a consequence of the Main Theorem. Since the Corollary deals with functions and forms on $\bar{\Omega}$ we will consider only the usual Lipschitz classes.

Proof of the Corollary. Since Ω is pseudoconvex and $\Omega \subset \mathbb{C}^n$ we know that the range of $\bar{\partial}_b$ is closed (see [BS, S1, K5]). Thus $\mathcal{M}$, the boundary of Ω, satisfies the hypotheses of the theorem. The equation $\bar{\partial} u = \alpha$ implies $\bar{\partial}_b u_b = \alpha_b$; thus from part (d) of the theorem we conclude that $u_b \in \mathrm{Lip}(s + 1/m - \varepsilon, P_0)$. The equation $\bar{\partial} u = \alpha$ in terms of the coordinates in $\mathbb{C}^n$ can be expressed by $u_{\bar{z}_j} = \alpha_j$ for $j = 1, ..., n$ where $\alpha = \sum \alpha_j\, d\bar{z}_j$. Then we have $\sum_j u_{z_j \bar{z}_j} = \sum \alpha_{j z_j}$ with $u = u_0$ on the boundary and we conclude that part (a) of the Corollary now follows from the classical regularity results on the Dirichlet problem.

Similarly, part (b) of the Corollary follows from the fact that $\bar{\partial} S_b(f) = 0$, hence $\Delta S_b(f) = 0$ and the restriction of $S_b(f)$ to the boundary equals $S_b(f_b)$ which is in $\mathrm{Lip}(s - \varepsilon, P_0)$ by part (c) of the Main Theorem.

0.11. *Remark.* In the Corollary $\mathbb{C}^N$ can be replaced by a complex analytic manifold with the property that there exists a strongly plurisubharmonic function defined in a neighborhood of the boundary of Ω. Thus, for example, all Stein manifolds have this property. It then follows (see [K5] for $n \geqslant 2$, [S1] for $n \geqslant 3$, and [BS] for $\Omega \subset \mathbb{C}^2$) that the range of $\bar{\partial}_b$ is closed and hence the argument given above holds.

0.12. *Remark.* If $\dim \mathcal{M} \geqslant 5$, if $\mathcal{M}$ is compact pseudoconvex, and if each point of $\mathcal{M}$ is of finite type then $\Box_b$ on (p, q)-forms with $1 \leqslant q \leqslant n-2$ is subelliptic. This implies that the range of $\bar{\partial}_b$ on forms of any type is closed. If $\dim \mathcal{M} = 3$ this no longer holds, see Rossi [R] and Burns [B].

Guide to the Proof of the Main Theorem

Here we will indicate what the strategy of our proof is. First we consider regularity in terms of Sobolev norms, we will sketch the arguments given in [K2] for the diagonalizable case.

To prove Sobolev regularity of $\Box_b^{-1}$ on (p, q)-forms it suffices to show (see [KN]) that there exists a neighborhood U of P_0 and positive constants ε, c such that

$$\|\varphi\|_\varepsilon^2 \leqslant c(\Box_b \varphi, \varphi) \tag{11}$$

for all (p, q)-forms whose coefficients are in $C_0^\infty(U)$. To simplify matters we will restrict ourselves to $(0, 1)$-forms. Let $L_1, ..., L_{n-1}, \omega^1, ..., \omega^{n-1}$ be bases as above then by integration by parts (see [K2]), we obtain

$$(\Box_b \varphi, \varphi) = \begin{cases} \sum_{j,k} \|\bar{L}_j \varphi_k\|^2 - \sum_j (\lambda_j \sqrt{-1}\, T\varphi_j, \varphi_j) \\ \quad + O(\sum \|\bar{L}_j \varphi_k\|\ \|\varphi\| + \|\varphi\|^2) \\ \sum_{j,k} \|L_j \varphi_k\|^2 + \sum_j ((\sum_{i \neq j} \lambda_i) \sqrt{-1}\, T\varphi_j, \varphi_j) \\ \quad + O(\sum \|L_j \varphi_k\|\ \|\varphi\| + \|\varphi\|^2). \end{cases} \tag{12}$$

For $u \in C_0^\infty(U)$ we have, from (1),

$$\|L_j u\|^2 = -(\lambda_j \sqrt{-1}\, Tu, u) + \|\bar{L}_j u\|^2 + O\left(\sum_k \|\bar{L}_k u\|\ \|u\| + \|u\|^2\right). \tag{13}$$

Furthermore, since L_j is of finite type, we have

$$\|u\|_\varepsilon^2 \leqslant c\left(\|L_j u\|^2 + \sum_k \|\bar{L}_k u\|^2\right) \tag{14}$$

for $u \in C_0^\infty(U)$, U small.

Thus for φ satisfying

$$-\sum_j (\lambda_j \sqrt{-1}\, T\varphi_j, \varphi_j) \geqslant -\text{const.}\ \|\varphi\|^2 \tag{15}$$

we have

$$\sum_j \|L_j \varphi_j\|^2 + \sum_{j,k} \|\bar{L}_k \varphi_j\|^2 \leqslant c(\Box_b \varphi, \varphi) \tag{16}$$

so that the desired estimate (11), for φ satisfying (15), follows from (14).

Similarly, if we have

$$\sum_j \left(\sum_{i \neq j} \lambda_i \sqrt{-1}\, T\varphi_j, \varphi_j\right) \geqslant -\text{const.}\ \|\varphi\|^2 \tag{17}$$

then

$$\sum_{j,k} \|L_k \varphi_j\|^2 + \sum_j \left(\sum_{i \neq j} \|\bar{L}_i \varphi_j\|^2\right) \leqslant c(\Box_b \varphi, \varphi) \tag{18}$$

and again (11) holds. We wish to decompose φ into a sum of three pieces, one which satisfies (11) with $\varepsilon = 1$ and the remaining two which satisfy (15) and (16), respectively. To do this we choose coordinates $x_0, x_1, ..., x_{2n-2}$ on U such that $T = \partial/\partial x_0$, to do this we can replace T by gT with $g \in C^\infty(U)$ and $g > 0$. Let $\mathscr{R}^+$ be a subset of $\mathbb{R}^{2n-1}$ consisting of those $\xi \in \mathbb{R}^{2n-1}$ such that $|\xi| > 1/2$ and $\xi_0 > c(\sum_{j=1}^{2n-1} \xi_j^2)^{1/2}$ and $\mathscr{R}^-$ consists of $\xi \in \mathbb{R}^{2n-1}$ such that $|\xi| > 1/2$ and $\xi_0 < -c\,(\sum_1^{2n-1} \xi_j^2)^{1/2}$, where c is a positive constant. Let $\mathscr{R}^0$ consist of $\xi \in \mathbb{R}^2$ such that either $|\xi_0| < c(\sum_{j=1}^{2n-1} \xi_j^2)^{1/2}$ or $|\xi| < 1$. Choosing C and c so that $\mathscr{R}^+ \cup \mathscr{R}^0 \cup \mathscr{R}^- = \mathbb{C}^n$, let $\chi^+, \chi^0, \chi^- \in C^\infty(\mathbb{R}^n)$ be a partition of unity, that is, $\chi^+ + \chi^0 + \chi^- = 1$ and $\operatorname{supp} \chi^+ \subset \mathscr{R}^+$, $\operatorname{supp} \chi^0 \subset \mathscr{R}^0$, and $\operatorname{supp} \chi^- \subset \mathscr{R}^-$. Suppose further that these functions are homogeneous or order 0 for $|\xi| \geqslant 1$, that is, $\chi(t\xi) = \chi(\xi)$ whenever $|\xi| \geqslant 1$ and $t \geqslant 1$. We define operators P^+, P^0, and P^- by

$$\widehat{Pu}(\xi) = \chi(\xi)\, \hat{u}(\xi). \tag{19}$$

So that if $\zeta = 1$ on $\operatorname{supp} u$, we have $\zeta P^+ u + \zeta P^0 u + \zeta P^- u = u$.

Since $\square_b$ is elliptic on the support of χ^0, we obtain

$$\|\zeta P^0 \varphi\|_1^2 \leqslant c\{(\zeta P^0 \square_b \varphi, \zeta P^0 \varphi) + \|\varphi\|^2\}. \tag{20}$$

The principal symbols of the operators $-P^+ \zeta \lambda_j \sqrt{-1}\; T\zeta P^+$ and $+P^- \zeta(\sum_{i \neq j} \lambda_i)\; \sqrt{-1}\; T\zeta P^-$ are $\zeta^2 \lambda_j \xi_0^+$ and $\zeta^2 \sum_{i \neq j} \lambda_j \xi_0^-$, respectively. Since these symbols are non-negative, we can apply the Garding inequality and obtain

$$-(\lambda_j \sqrt{-1}\, \zeta T P^+ \varphi_j, \zeta P^+ \varphi_j) \geqslant -\text{const.}\, \|\varphi_j\|^2 \tag{21}$$

and

$$\left(\left(\sum_{i \neq j} \lambda_i\right) \sqrt{-1}\; T\zeta P^- \varphi_j, \zeta P^- \varphi_j\right) \geqslant -\text{const.}\, \|\varphi_j\|^2, \tag{22}$$

where $\zeta \in C_0^\infty(\mathbb{R}^{2n-1})$ and $\zeta = 1$ in a neighborhood of U. Reasoning as above and using integration by parts to absorb on the left the terms that arise from the commutators $[\zeta P^+, \square_b]$ and $[\zeta P^-, \square_b]$, we obtain

$$\sum_j \|L_j \zeta P^+ \varphi_j\|^2 + \sum_{j,k} \|\bar{L}_k \zeta P^+ \varphi_j\|^2 \leqslant C[(\zeta P^+ \square_b \varphi, \zeta P^+ \varphi) + \|\varphi\|^2] \tag{23}$$

and

$$\sum_{j,k} \|L_k \zeta P^- \varphi_j\|^2 + \sum_j \left(\sum_{i \neq j} \|\bar{L}_i \zeta P^- \varphi_j\|^2\right)$$
$$\leqslant C[(\zeta P^- \square_b \varphi, \zeta P^- \varphi) + \|\varphi\|^2]. \tag{24}$$

Now applying (14) to (23) and (24) and combining this with (10), we obtained the desired estimate (11).

To prove the main theorem we will reduce the study of the inverse of $\Box_b$ to the study of inverses of certain real pseudodifferential operators. First we will try to reduce the study of $\Box_b$ to the study of the diagonal matrix $\Box_d$. On U we set $\Box_b = \Box_d + \mathscr{L}$, where

$$\Box_d = \begin{pmatrix} \Box_1 & & & 0 \\ & \Box_2 & & \\ & & \ddots & \\ 0 & & & \Box_{n-1} \end{pmatrix}, \tag{25}$$

$$\Box_i = -\sum_j L_j \bar{L}_j + [L_i, \bar{L}_i], \tag{26}$$

and $\mathscr{L} = (\mathscr{L}_{ij})$ with $\mathscr{L}_{ij}$ of the form

$$\mathscr{L}_{ij} = \sum_k a_{ij}^k L_k + \sum b_{ij}^k \bar{L}_k + e_{ij}. \tag{27}$$

The fact that the L_i diagonalize the Levi form implies that T does not appear in (27). The proof of (11) shows that the operators $\Box_i$ are subelliptic and invertible. In fact the derivation of the estimates (23) and (24) shows that, for purposes of local Sobolev estimates, the operators $\Box_b$ and $\Box_d$ have the same properties. This is because the error terms introduced by $(\mathscr{L}_{ij})$ are of the form $(L\varphi, \varphi)$ and $(\bar{L}\varphi, \varphi)$ *both* of which can be estimated by $\|\bar{L}\varphi\| \, \|\varphi\| + \|\varphi\|^2$; thus these terms can be controlled by (23) when φ is replaced by $\zeta P^+ \varphi$. These error terms can also be estimated by $\|L\varphi\| \, \|\varphi\| + \|\varphi\|^2$ and thus controlled by (24) when φ is replaced by $\zeta P^- \varphi$. For purposes of Hölder estimates we cannot handle the errors introduced by $\mathscr{L}$ in this way. We will use the formula

$$\Box_b^{-1} = \Box_d^{-1} - \Box_d^{-1} \mathscr{L} \Box_d^{-1} + \cdots + (-1)^{k+1} \underbrace{\Box_d^{-1} \mathscr{L} \cdots \Box_d^{-1} \mathscr{L}}_{2k \text{ terms}} \Box_b^{-1}. \tag{28}$$

We want to prove that

$$\underbrace{\Box_d^{-1} \mathscr{L} \Box_d^{-1} \cdots \mathscr{L} \Box_d^{-1}}_{2k-1 \text{ terms}}$$

maps $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s + k/m - \varepsilon, P_0)$ and that

$$\underbrace{\Box_1^{-1} \mathscr{L} \cdots \Box_d^{-1} \mathscr{L}}_{2k \text{ terms}}$$

maps $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s + (k+1)/m - 1 - \varepsilon, P_0)$. From this it follows, by taking k large enough, that if $\square_d^{-1}$ maps $\mathrm{Lip}(s, P_0)$ to $\mathrm{Lip}(s+2/m-\varepsilon, P_0)$ then so does $\square_b^{-1}$. To prove this we will prove the following.

LEMMA. *Let j, $k \in \{1, ..., n-1\}$. Setting $(\square_i^{-1})^+ = \square_i^{-1}\zeta P^+$ and $(\square_i^{-1})^- = \square_i^{-1}\zeta P^-$ we have that $(\square_i^{-1})^+$ maps $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s+2/m-\varepsilon, P_0)$; $\bar{L}_j(\square_i^{-1})^+$, $(\square_i^{-1})^+ L_j$, $L_i(\square_i^{-1})^+$, $(\square_i^{-1})^+ \bar{L}_i$ map $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s+1/m-\varepsilon, P_0)$; the following operators map $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s-\varepsilon, P_0)$: $\bar{L}_j(\square_i^{-1})^+ L_k$, $\bar{L}_j(\square_i^{-1})^+ \bar{L}_i$, $L_i(\square_i^{-1})^+ L_j$; $L_i(\square_i^{-1})^+ \bar{L}_i$. For the operators $(\square_i^{-1})^-$ similar mapping properties hold exchanging the L's with the $\bar{L}$'s and the set $\{i\}$ with its complement.*

Remark. Since the $\square_i$ are elliptic on the region $\mathscr{R}^0$ the operators $(\square_i^{-1})^0 = \square_i^{-1}\zeta P^0$ satisfy all the above properties with $m=1$.

To see that the above lemma implies that terms $\square_d^{-1}\mathscr{L} \cdots \mathscr{L}\square_d^{-1}$ have the mapping properties claimed above, we will consider an element of the form $\square_1^{-1}\bar{L}_2\square_3^{-1}L_1\square_2^{-1}$ in the matrix $\square_d^{-1}\mathscr{L}\square_d^{-1}\mathscr{L}\square_d^{-1}$. We have

$$\square_1^{-1}\bar{L}_2\square_3^{-1}L_1\square_2^{-1}\zeta P^+ \sim [(\square_1^{-1})^+][\bar{L}_2(\square_3^{-1})^+ L_1][(\square_2^{-1})^+],$$

by the lemma the first and last term map $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s+2/m-\varepsilon, P_0)$ and the middle term maps $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s-\varepsilon, P_0)$ so that the operator maps $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s+4/m-\varepsilon, P_0)$. Similarly, writing

$$\square_1^{-1}\bar{L}_2\square_3^{-1}L_1\square_2^{-1}\zeta P^- \sim [(\square_1^{-1})^- \bar{L}_2][(\square_3^{-1})^-][L_1(\square_2^{-1})^-]$$

the first and last term improve Lipschitz regularity by $1/m-\varepsilon$ each and the middle term by $2/m-\varepsilon$, again gaining the desired $4/m-\varepsilon$ improvement (here $\sim$ indicates that the difference is of order $-\infty$). In this way we show that (a) of the main theorem follows from the lemma. Arguing along these lines, one proves that parts (b), (c), (d), and (e) also follow from the lemma and from (28). Although it would seem that a term such as $\bar{\partial}_b\bar{\partial}_b^*\square_b^{-1}$ cannot be analyzed by our lemma, it can, since it equals $\bar{\partial}_b\square_b^{-1}\bar{\partial}_b^*$.

To prove the lemma we treat the operators

$$\square_{i_1 \cdots i_q} = -\sum_j L_j\bar{L}_j + \sum_k [L_{i_k}, \bar{L}_{i_k}], \qquad q=1, ..., n-2 \tag{29}$$

by induction on q to prove the mapping properties of $(\square_{i_1 \cdots i_q}^{-1})^-$ and induction on $s=n-1-q$ to prove the mapping properties of $(\square_{i_1 \cdots i_q}^{-1})^+$. The operators $\square_{i_1 \cdots i_q}$ arise in the expression for $\square_b$ on $(0, q)$-forms. Here we will only discuss the case when $n=3$, $q=1$, i.e., $(0, 1)$-forms on $\mathscr{M}$ with $\dim \mathscr{M} = 5$. In this case we have the operators

$$\square_1 = -\bar{L}_1L_1 - L_2\bar{L}_2 \qquad \text{and} \qquad \square_2 = -L_1\bar{L}_1 - \bar{L}_2L_2. \tag{30}$$

We will restrict ourselves to $(\square_1^{-1})^+$. The corresponding results for $(\square_1^{-1})^-$, $(\square_2^{-1})^+$, and $(\square_2^{-1})^-$ then will follow by changes of notation.

First we observe that it suffices to establish the lemma for functions with compact support. We suppose that $\operatorname{supp} u \subset U$ and that $\zeta \in C_0^\infty(\mathbb{R}^n)$ equals 1 on U.

Set $\square_1^+ u = \square_1 P^+ u$, then we have

$$\square_1^+ u = \zeta P^+ \square_1 u + [\square_1, \zeta]\, P^+ u + \zeta[\square_1, P^+]u. \tag{31}$$

The second and third term on the right can be written in the form

$$\sum_1^{2n-2} Z_j \tilde{\zeta}_j \tilde{P}^+ u + R^{-\infty} u,$$

where the Z_j are combinations of $L_1, ..., L_{n-1}$, $\bar{L}_1, ..., \bar{L}_{n-1}$, $\tilde{\zeta}_j \in C_0^\infty$ and $\tilde{\zeta}_j = 1$ on U, $\tilde{P}^+$ is a pseudodifferential operator of order zero whose symbol is supported in a region of type $\mathscr{R}^+$, and $R^{-\infty}$ is an infinitely smoothing pseudodifferential operator. These terms are "lower order" in the sense that their inner product with u can be estimated. Thus we can replace $(\square_1^{-1})^+ = \square_1^{-1}\zeta P$ by $\zeta P^+ \square_1^{-1}$. Denoting by $\mathscr{C}^+$ the space of functions of the form $\zeta P^+ u$, we will say that $\square_1^+$ is the restriction of $\square_1$ to $\mathscr{C}^+$ and $(\square_1^{-1})^+$ is the restriction of $\square_1^{-1}$ to $\mathscr{C}^+$. We will study the operator $\square_1^+$ by means of operators of the form

$$A = -\sum X_j \circ \varphi_j \circ X_j + \lambda\, |T|, \tag{32}$$

where $\lambda \geqslant 0$, and $|T|$ is a pseudodifferential operator of order one whose symbol is smooth and on the region $\mathscr{R}^+ \cup \mathscr{R}^-$ is equal to $|\xi_0|$. The X_j are real vector fields, the $\varphi_j \geqslant 0$ and these are defined on a neighborhood of P_0.

We require also that the operator $\sum X_j \circ \varphi_j \circ X_j$ is subelliptic, such operators were studied in [RS] when the $\varphi_j = 1$ and in general in [FS]. One of the major purposes of this paper is to show that, under suitable restrictions on λ, the operator A^{-1} and the operator $(-\sum X_j \circ \varphi_j \circ X_j)^{-1}$ have the same mapping properties which are as follows. A^{-1} maps $\mathrm{Lip}(s, P_0)$ to $\mathrm{Lip}(s + 2/m, P_0)$, where m is related to the subellipticity of $-\sum X_j \varphi_j X_j$ by

$$\|u\|_{1/m}^2 \leqslant C\left(\sum (\varphi_j X_j u, X_j u) + \|u\|^2\right). \tag{33}$$

We define a *subunit vector field* Z to be a combination of vectors of the form $h_j X_j$ with $|h_j|^2 \leqslant \varphi_j$. Then we prove ZA^{-1}, $A^{-1}Z$ map $\mathrm{Lip}(s, P_0)$ to $\mathrm{Lip}(s + 1/m, P_0)$ when Z is subunit. Furthermore, that $ZZ'A^{-1}$, $ZA^{-1}Z'$, AZZ' map $\mathrm{Lip}(s, P_0)$ to $\mathrm{Lip}(s - \varepsilon, P_0)$ when Z and Z' are subunit. In our proof we will also need operators of the form e^{-tA} and in particular we will use the representation

$$A^{-1} \sim \int_0^{t_0} e^{-tA}\, dt + \text{smoothing operator}, \tag{34}$$

with $t_0 > 0$.

We would like to write $\Box_1^+$ as the restriction to $\mathscr{C}^+$ of an operator A in the form (32) modulo combinations of subunit vectors and smoothing operators. This is not possible, since $\|L_2 \zeta P^+ u\|$ is not dominated by $\|\Box_1^+ u\|$. So, since $\|L_1 \zeta P^+ u\|$ and $\|\bar{L}_1 \zeta P^+ u\|$ are dominated by $\|\Box_1^+ u\|$, we will construct an operator A_1 of the form (32) for which $\mathrm{Re}(L_1)$ and $\mathrm{Im}(L_1)$ are subunit vectors and such that

$$|(A_1^+ u, u)| \leqslant \text{const.}\{|(\Box_1^+ u, u)| + \|u\|^2\}.$$

Note that on $\mathscr{C}^+$ we have $\sqrt{-1}\, T \sim |T|$. Since

$$[L_1, \bar{L}_1] = \lambda_1 \sqrt{-1}\, T + aL_2 - \bar{a}\bar{L}_2 \qquad \mathrm{mod}(L_1, \bar{L}_1)$$

we have, on $\mathscr{C}^+$

$$-\bar{L}_1 L_1 - L_2 \circ g \circ \bar{L}_2 \sim \mathrm{Re}(\bar{L}_1 L_1 + L_2 \circ g \circ \bar{L}_2) - aL_2$$
$$+ \bar{a}\bar{L}_2 + \tfrac{1}{2}(\lambda_1 - g\lambda_2)\, |T| \qquad \mathrm{mod}(L_1, \bar{L}_1). \tag{35}$$

Thus we want to choose $g \geqslant 0$, so that the operator $-\mathrm{Re}(\bar{L}_1 L_1 + L_2 \circ g \circ \bar{L}_2)$ is subelliptic with $\mathrm{Re}(aL_2)$, $\mathrm{Im}(aL_2)$ subunit vectors. Note that $\mathrm{Re}(L_1)$ and $\mathrm{Im}(L_1)$ are automatically subunit vectors. The operators $-\mathrm{Re}(\bar{L}_1 L_1 + L_2 \circ g \circ \bar{L}_2)$ is subelliptic if there exists some polynomial p in L_1 and $\bar{L}_1$ that $p(L_1, \bar{L}_1)\, g(P_0) \neq 0$. A function g satisfying these conditions is

$$g = \lambda_1 + c_1\, |a|^2 \tag{36}$$

with $c_1 > 0$.

For A_1 to be in the form given by (32), we also require that $\lambda_1 - g\lambda_2 \geqslant 0$, this holds if and only if there exists $C_0 > 0$ such that

$$\lambda_2\, |a|^2 \leqslant C_0 \lambda_1. \tag{37}$$

We will assume for the moment that (37) holds, later we will indicate the modifications in the argument that are needed to drop this assumption.

From (35) we see that on $\mathscr{C}^+$ we have

$$\Box_1 \sim A_1 - L_2 \circ (1 - g) \circ \bar{L}_2 + Z,$$

where Z is a combination of subunit vectors for A_1. Hence

$$\Box_1^{-1} \sim A_1^{-1} + A_1^{-1} L_2 \circ (1-g) \circ \bar{L}_2 \Box_1^{-1} - A_1^{-1} Z \Box_1^{-1}. \tag{38}$$

Thus if $\zeta P^+ u$, $(\Box_1^{-1})^+ u$, $\bar{L}_2(\Box_1^{-1})^+ u$, and $L_2\bar{L}_2(\Box_1^{-1})^+ u$ are in $\mathrm{Lip}(s, P_0)$ and W, W' are subunit for A_1, then

$$(\Box_1^{-1})^+ u \in \mathrm{Lip}\left(s + \frac{2}{m}; P_0\right);$$

$$W(\Box_1^{-1})^+ u \text{ and } (\Box_1^{-1})^+ Wu \in \mathrm{Lip}\left(s + \frac{1}{m}; P_0\right)$$

$$WW'(\Box_1^{-1})^+ u \text{ and } (\Box_1^{-1})^+ WW'u \in \mathrm{Lip}(s-\varepsilon, P_0).$$

Next, consider the operator $-\mathrm{Re}(\bar{L}_1 L_1 + \bar{L}_2 L_2)$. This operator itself is of the form $-\sum X_i^2$, where the X_i are real vector fields that satisfy Hörmander's condition of order $m_0 = \min(m_1, m_2)$. Let A_0 be given by

$$A_0 = -\mathrm{Re}(\bar{L}_1 L_1 + \bar{L}_2 L_2) + \tfrac{1}{2}(\lambda_1 + \lambda_2)\,|T| \tag{39}$$

then on $\mathscr{C}^+$ we have

$$A_0 \sim -\bar{L}_1 L_1 - \bar{L}_2 L_2 \tag{40}$$

hence

$$\begin{aligned} \bar{L}_2 \Box_1 &\sim A_0 \bar{L}_2 - [\bar{L}_2, \bar{L}_1 L_1] \\ &\sim A_0 \bar{L}_2 - Z L_1 - Z' \bar{L}_1 - Z'', \end{aligned}$$

where Z, Z', and Z'' are combinations of L_1, $\bar{L}_1$, L_2, $\bar{L}_2$ and hence also combinations of subunits for A_0. Then on $\mathscr{C}^+$

$$\bar{L}_2 \Box_1^{-1} \sim A_0^{-1} \bar{L}_2 + A_0^{-1} Z L_1 \Box_1^{-1} + A_0^{-1} Z' \bar{L}_1 \Box_1^{-1} + A_0^{-1} Z'' \Box_1^{-1}. \tag{41}$$

Now if $\zeta P^+ u \in \mathrm{Lip}(s, P_0)$ we can assume that for some $s_0 > 0$ we have $(\Box_1^{-1})^+ u$, $\bar{L}_2(\Box_1^{-1})^+ u$, and $L_2\bar{L}_2(\Box_1^{-1})^+ u \in \mathrm{Lip}(-s_0, P_0)$. Then from (3.8) we conclude that $(\Box_1^{-1})^+ u \in \mathrm{Lip}(-s_0 + 2/m, P_0)$; $L_1(\Box_1^+)^{-1} u$, $\bar{L}_1(\Box_1^+)^{-1} u \in \mathrm{Lip}(s_0 + 1/m, P_0)$. Applying (41), we then conclude that $\bar{L}_2(\Box_1^{-1})^+ u \in \mathrm{Lip}(-s_0 + 2/m, P_0)$ and that $L_2\bar{L}_2(\Box_1^{-1})^+ u \in \mathrm{Lip}(-s_0 + 1/m - \varepsilon, P_0)$. Continuing to apply (38) and (41) successively j-times with j such that $-s_0 + j/m \leqslant s \leqslant -s_0 + (j+1)/m$, we obtain

$(\Box_1^{-1})^+ u \in \mathrm{Lip}(s+2/m-\varepsilon. P_0)$; $L_1(\Box_1^{-1})^+ u$, $\bar{L}_1(\Box_1^{-1})^+ u$, $\bar{L}_2(\Box_1^{-1})^+ \in \mathrm{Lip}(s+1/m-\varepsilon, P_0)$; $L_2\bar{L}_2(\Box_1^{-1})^+ u \in \mathrm{Lip}(s-\varepsilon, P_0)$. Applying L_1^2, $\bar{L}_1^2$, $L_1\bar{L}_1$, and $\bar{L}_1 L_1$ to (38) and $Z=L_1$, $\bar{L}_1$, L_2, $\bar{L}_2$ to (41), we obtain $L_1^2(\Box_1^{-1})^+ u$, $\bar{L}_1^2(\Box_1^{-1})^+ u$, L_1, $\bar{L}_1(\Box_1^{-1})^+ u$, $Z\bar{L}_2(\Box_1^{-1})^+ u \in \mathrm{Lip}(s-\varepsilon, P_0)$. Taking adjoints, we obtain $(\Box_1^{-1})^+ \bar{L}_1 u$, $(\Box_1^{-1})^+ L_1 u$, $(\Box_1^{-1})^+ L_2 u \in \mathrm{Lip}(s + 1/m - \varepsilon, P_0)$, $(\Box_1^{-1})^+ \bar{L}_1^2 u$, $(\Box_1^{-1})^+ L_1^2 u$, $(\Box_1^{-1})^+ \bar{L}_1 L_1$, $(\Box_1^{-1})^+ \bar{L}_1 L_1$, $(\Box_1^{-1})^+ L_2 Z \in \mathrm{Lip}(s-\varepsilon, P_0)$. To prove that $Z(\Box_1^{-1})^+ Wu \in \mathrm{Lip}(s-\varepsilon, P_0)$ with Z and W as in the lemma, we study the operator $e^{-t(\Box_1)^+}$; we give a brief indication of our approach below.

For operators A in form (32), we prove estimates of the form

$$\|\Gamma_\delta e^{-tA} Yu\|_{L^\infty} + \|\Gamma_\delta Y e^{-tA} u\|_{L^\infty} \leqslant C \min(t^{1/2}, \delta^{-100} \ln |t|)(\|\tilde{\Gamma}_\delta u\|_{L^\infty} + C\delta^p \|u\|_{-s_0}), \tag{42}$$

where Y is a subunit for A, $\tilde{\Gamma}_\delta$ denotes multiplication of the Fourier transform by a function $\tilde{\psi}((\delta/M)|\xi|)$ with $\tilde{\psi} \in C_0^\infty(\{\xi \in \mathbb{R}^{2n-1} | 0 < \tilde{a} < |\xi| < \tilde{b}\})$, and $\psi = 1$ on a neighborhood of supp ψ (see Definition 9), p and s_0 are arbitrarily large. From these estimates for A_1 and A_0 we prove that the L^∞-norms of $\Gamma_\delta Z e^{-t\Box_1^+} u$ and $\Gamma_\delta e^{-t\Box_1^+} Wu$ are bounded by the right-hand side of (42) whenever $Z = L_1$, $\bar{L}_1$, or $\bar{L}_2$ and $W = L_1$, $\bar{L}_1$, or L_2. To prove this we follow the analogous procedure as above to translate estimates for A_0 and A_1 to estimates for $\Box_1^+$. The desired estimate for $\Gamma_\delta Z(\Box_1^{-1})^+ Wu$ is then obtained by noting that

$$\Gamma_\delta Z e^{-t\Box_1^+} \circ \Gamma_\delta e^{-t\Box_1^+} W \sim \Gamma_\delta Z e^{-2t\Box_1^+} W, \tag{43}$$

integrating the corresponding estimates with respect to t and applying (34). In our analysis of the operator $e^{-t\Box_1}$ we adopt the formalism used by Stanton and Tartakoff (see [ST]) in their study of the operator $e^{-t\Box_b}$ on strongly pseudoconvex CR manifolds.

To drop the assumption that (37) holds, we prove the weaker inequality

$$\lambda_2 |a|^2 \leqslant C(\lambda_1 \delta^{-\varepsilon} + \delta^N) \tag{44}$$

which holds for any $\varepsilon > 0$ and N large. This enables us to define an operator $\tilde{A}_1$, depending on δ, which is used in the same way as was A_1 and through which we obtain the desired estimates.

Remark. For pseudoconvex CR manifolds of finite type of dimension 5 we can prove, even if the Levi form is not smoothly diagonalizable, that $\bar{\partial}_b^* \bar{\partial}_b \Box_b^{-1}$ on (0, 1) forms maps $\mathrm{Lip}(s, P_0)$ into $\mathrm{Lip}(s-\varepsilon, P_0)$. Indeed, $\bar{\partial}_b \Box_b^{-1} \sim \Box_b^{-1} \bar{\partial}_b \sim A_0^{-1} \bar{\partial}_b$ on $\mathscr{C}^+$ and $\bar{\partial}_b^* \Box_b^{-1} \sim A_0^{-1} \bar{\partial}_b^*$ on $\mathscr{C}^-$. This is closely related to (41).

PART I: THE OPERATOR e^{-tA}

1. SET-UP

Let X_i be smooth real vector fields on a compact manifold $\mathcal{M}$ without boundary, and let ϕ_i be non-negative smooth functions on $\mathcal{M}$. Form the second-order operator $L = \sum_i X_i^* \phi_i X_i$, and suppose L is subelliptic. Let $B_L(x, \gamma)$ be the non-Euclidean balls associated to L, and define $\gamma(x, \delta)$ to be the value of γ for which $B_L(x, \gamma)$ has shortest dimension δ. Let $\lambda(x)$ be a smooth function on $\mathcal{M}$ satisfying the estimate $|\lambda(x)| \leqslant C\,\delta\gamma^{-2}(x, \delta)$ for $x \in \mathcal{M}$, $0 < \delta < 1$. Let Q^1 and Q^0 be pseudodifferential operators of order 1 and 0 on $\mathcal{M}$, and define $S = \lambda(x) Q^1 + Q^0$. Assume S has non-negative principal symbol and is self-adjoint.

The object of our study is the pseudodifferential operator $A = L + S$, and in particular the semigroup e^{-sA}, $0 < s < \infty$.

To measure the smoothness of functions, introduce pseudodifferential operators of order zero Γ_δ, $\tilde{\Gamma}_\delta$ on $\mathcal{M}$, with symbol $\tilde{\Gamma}_\delta = 1$ on the support of the symbol of Γ_δ, and with Γ_δ, $\tilde{\Gamma}_\delta$ having symbols supported in $\{(x, \xi) \in T^*\mathcal{M} \mid |\xi| \sim 1/\delta\}$ $(0 < \delta < 1/2)$.

THEOREM (1.1). *Let Y, Y' be smooth vector fields on $\mathcal{M}$, subunit with respect to L. Then we have the estimates*

$$\|\Gamma_\delta e^{-tA} f\|_{L^\infty} \leqslant C\,|\ln t|\,(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})$$

$$\|\Gamma_\delta Y e^{-tA} f\|_{L^\infty} \leqslant C \min(t^{-1/2}, \delta^{-100}\,|\ln t|)(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})$$

$$\|\Gamma_\delta e^{-tA} Y f\|_{L^\infty} \leqslant C \min(t^{-1/2}, \delta^{-100}\,|\ln t|)(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})$$

$$\|\Gamma_\delta Y Y' e^{-tA} f\|_{L^\infty} \leqslant C\,|\ln \delta| \min(t^{-1}, \delta^{-100}\,|\ln t|)(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})$$

$$\|\Gamma_\delta Y e^{-tA} Y' f\|_{L^\infty} \leqslant C\,|\ln \delta| \min(t^{-1}, \delta^{-100}\,|\ln t|)(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})$$

$$\|\Gamma_\delta e^{-tA} Y Y' f\|_{L^\infty} \leqslant C\,|\ln \delta| \min(t^{-1}, \delta^{-100}\,|\ln t|)(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}).$$

In these estimates, p and s_0 are arbitrarily large. The constants C depend on p, s_0, but not on f.

The proof goes as follows.

2. ELEMENTARY REMARKS

1. Suppose $P_1 \cdots P_n$ are pseudodifferential operators. Suppose P_i and P_j have symbols with disjoint supports, and that the symbol of P_i is supported in $\{(x, \xi) \in T^*\mathcal{M} \mid |\xi| > 1/\delta\}$. Then

$$\|P_1 \cdots P_n f\|_{s_0} \leqslant C_{p, s_0}\,\delta^p\,\|f\|_{-s_0} \qquad \text{for any} \quad p, s_0.$$

2. We will often reason as follows: Suppose

$$\|\Gamma_\delta Au\|_{L^\infty} \leqslant C\,\|\tilde{\Gamma}_\delta\, Bu\|_{L^\infty} + C\,\delta^p\,\|u\|_{-s_0} \tag{1}$$

$$\|\Gamma_\delta Bu\|_{L^\infty} \leqslant C\,\|\tilde{\Gamma}_\delta\, Su\|_{L^\infty} + C\,\delta^p\,\|u\|_{-s_0} \tag{2}$$

for operators A, B, S (say). These estimates are to hold for any Γ_δ, $\tilde{\Gamma}_\delta$ with symbols supported in $\{|\xi| \sim 1/\delta\}$, with $\tilde{\Gamma}_\delta = 1$ on support (Γ_δ).

Let Γ_δ, $\tilde{\Gamma}_\delta$, $\tilde{\tilde{\Gamma}}_\delta$ be symbols supported in successively larger regions. From (2) we obtain as a special case

$$\|\tilde{\Gamma}_\delta Bu\|_{L^\infty} \leqslant C\,\|\tilde{\tilde{\Gamma}}_\delta Su\|_{L^\infty} + C\,\delta^p\,\|u\|_{-s_0}.$$

Combining this with (1), we find that

$$\|\Gamma_\delta Au\|_{L^\infty} \leqslant C\,\|\tilde{\tilde{\Gamma}}_\delta Su\|_{L^\infty} + C\,\delta^p\,\|u\|_{-s_0}. \tag{3}$$

This can be established for any $\tilde{\tilde{\Gamma}}_\delta = 1$ on supp Γ_δ, since we can concoct an appropriate intermediate $\tilde{\Gamma}_\delta$. Hence as a special case of (3) we obtain

$$\|\Gamma_\delta Au\|_{L^\infty} \leqslant C\,\|\tilde{\Gamma}_\delta Su\|_{L^\infty} + C\,\delta^p\,\|u\|_{-s_0}. \tag{4}$$

Thus, estimates like (1) and (2) can be composed to yield (4).

We use this idea often without explicitly switching Γ, $\tilde{\Gamma}$, $\tilde{\tilde{\Gamma}}$.

3. Geometry

Let $L = -\sum_{ij} \partial_i a_{ij}\, \partial_j$ be an operator on $\mathbb{R}^n$ with smooth real coefficients. Assume $(a_{ij}(x)) \geqslant 0$ everywhere, and suppose L is subelliptic. Let $B_L(x, \gamma)$ be the non-Euclidean ball associated to x, γ, L; and define dist(x, y) to be the non-Euclidean distance. There is a "straightening" map Φ: $\prod_{k=1}^n I_k^+ \to \mathbb{R}^n$ with the properties

$$I_k^+ = \{\bar{x}_k \in \mathbb{R}^1 \mid |\bar{x}_k| \leqslant \gamma^{-\varepsilon}\,\delta_k\}, \qquad I_k = \{\bar{x}_k \in \mathbb{R}^1 \mid |\bar{x}_k| \leqslant \delta_k\}$$

$$B_L(x, c\gamma) \subset \Phi\left(\prod_{k=1}^n I_k\right) \subset B_L(x, C\gamma)$$

$$B_L(x, ct^\varepsilon\gamma) \subset \Phi\left(\prod_{k=1}^n \{|\bar{x}_k| \leqslant t\,\delta_k\}\right) \subset B_L(x, Ct^K\gamma) \qquad \text{for} \quad 1 \leqslant t \leqslant \gamma^{-\varepsilon}$$

$$\left|\left(\frac{\partial}{\partial \bar{x}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \bar{x}_n}\right)^{\alpha_n} \nabla\Phi\right| \leqslant C_{\alpha_1 \cdots \alpha_n} \prod_{k=1}^n (\gamma^{-\varepsilon}\delta_k)^{-\alpha_k} \qquad \text{on} \quad \prod_{k=1}^n I_k^+$$

$$\det(\nabla\Phi) = 1 \qquad \text{on} \quad \prod_{k=1}^n I_k^+.$$

The $\bar{x}_k$ are "straightened coordinates." Define a map $\tilde{\Phi}$: $\prod_{k=1}^n \{|\tilde{x}_k| \leqslant \gamma^{-\varepsilon}\} \to \mathbb{R}^n$ by

$$\tilde{\Phi}(\tilde{x}_1 \cdots \tilde{x}_n) = \Phi(\delta_1 \tilde{x}_1, ..., \delta_n \tilde{x}_n).$$

The $\tilde{x}_k$ are "rescaled, straightened coordinates."

Next we pull $\gamma^2 L$ back to rescaled, straightened coordinates by setting $\tilde{L}\tilde{u} = \tilde{v}$ for $\tilde{u} = u \circ \tilde{\Phi}$, $\tilde{v} = v \circ \tilde{\Phi}$, $v = \gamma^2 Lu$. We have $\tilde{L} = -\sum_{ij} (\partial/\partial\tilde{x}_i)\, \tilde{a}_{ij}(\partial/\partial\tilde{x}_j) + \sum_i \tilde{b}_i(\partial/\partial\tilde{x}_i) + \tilde{c}$ on the unit cube with coefficients $(\tilde{a}_{ij}) \geqslant 0$, $\tilde{b}_i$, $\tilde{c}$ smooth, and having their C^∞ seminorms bounded uniformly in x, γ. Moreover, a subelliptic estimate holds for $\tilde{L}$, uniformly in x, γ. Define $\delta(x, \gamma)$ to be the Euclidean distance from x to the complement of $B_L(x, \gamma)$, and define $\gamma(x, \delta)$ as the solution γ to the equation $\delta(x, \gamma) = \delta$. In terms of the straightening Φ, we have $\delta(x, \gamma) \sim \min_k(\delta_k)$. The properties of Φ just given imply the basic facts

$$ct^\varepsilon\, \delta(x, \gamma) \leqslant \delta(x, t\gamma) \leqslant Ct^K\, \delta(x, \gamma) \qquad \text{for} \quad 1 \leqslant t \leqslant \gamma^{-\varepsilon}$$

and therefore

$$ct^\varepsilon \gamma(x, \delta) \leqslant \gamma(x, t\delta) \leqslant Ct^K \gamma(x, \delta) \qquad \text{for} \quad 1 \leqslant t \leqslant \delta^{-\varepsilon}.$$

If $\Phi(\bar{y}_1, ..., \bar{y}_n) = y$, where Φ comes from $B_L(x, \gamma)$ as above, then

$$c\left(1 + \sum_{k=1}^n \frac{|\bar{y}_k|}{\delta_k}\right)^\varepsilon \leqslant 1 + \frac{\operatorname{dist}(x, y)}{\gamma} \leqslant C\left(1 + \sum_{k=1}^n \frac{|\bar{y}_k|}{\delta_k}\right)^K.$$

LEMMA 1.

$$\gamma(y, \delta) \leqslant C\gamma(x, \delta) \cdot \left(1 + \frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right) \tag{a}$$

$$\gamma(y, \delta) \geqslant c\gamma(x, \delta) \cdot \left(1 + \frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-K}. \tag{b}$$

Proof. (a) $B_L(x, \gamma_0)$ has shortest side $\sim\delta$, where $\gamma_0 = \gamma(x, \delta)$. Hence $B_L(x, \gamma_0 + \operatorname{dist}(x, y))$ has shortest side $\geqslant c\delta$. Since y belongs to this ball, it follows that $B_L(y, \gamma_0 + \operatorname{dist}(x, y))$ has shortest side $\geqslant c\delta$. Hence $\gamma(y, \delta) \leqslant C \cdot (\gamma_0 + \operatorname{dist}(x, y)) = C\gamma(x, \delta) \cdot (1 + \operatorname{dist}(x, y)/\gamma(x, \delta))$.

(b) Set $T = 1 + \operatorname{dist}(x, y)/\gamma(x, y)$, $\gamma_0 = \gamma(x, \delta)$. Since $B_L(x, \gamma_0)$ has shortest side $\sim\delta$, the ball $B_L(x, T\gamma_0)$ has shortest side $\leqslant CT^{K_1}\delta$. Since y belongs to this ball, $B_L(y, T\gamma_0)$ also has shortest side $\leqslant CT^{K_1}\delta$. Therefore, with $0 < t < 1$ to be picked, $B_L(y, tT\gamma_0)$ has shortest side $\leqslant C't^\varepsilon T^{K_1}\delta$. If $t = cT^{-(K_1/\varepsilon)}$ then we see that $B_L(y, cT^{-K}\gamma_0)$ has shortest side $\leqslant\delta$. Hence $\gamma(y, \delta) \geqslant cT^{-K}\gamma_0$ which is (b). ∎

COROLLARY.

$$c\left(1+\frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-K} \leqslant \frac{D+\gamma^{-2}(y, \delta)}{D+\gamma^{-2}(x, \delta)} \leqslant C\left(1+\frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{K}$$

for any constant $D \geqslant 0$, with c, C, K independent of D.

Proof. Trivial from (a) and (b). ▮

4. NORMS

Define (with M a fixed large constant to be picked later)

$$\|u\|_{\delta, \lambda, s}^{2}=\sup_{x}\left\{(\gamma^{-1}(x, \delta)+M^{-\varepsilon}|\lambda|)^{s} \times \int_{B_L(x, \gamma(x, \delta))}|u(y)|^{2}\, dy \Big/ \operatorname{vol} B_{L}(x, \gamma(x, \delta))\right\}$$

$$|||u|||_{\delta, \lambda, s}=\sup_{x}\left\{\frac{|u(x)|}{(\gamma^{-2}(x, \delta)+M^{-\varepsilon}|\lambda|)^{-s/2}}\right\}.$$

Obviously

$$\|u\|_{\delta, \lambda, s} \leqslant |||u|||_{\delta, \lambda, s},$$

$$|u(x)| \leqslant |||u|||_{\delta, \lambda, s} \cdot (\gamma^{-2}(x, \delta)+M^{-\varepsilon}|\lambda|)^{-s/2}$$

$$\int_{B_L(x, \gamma(x, \delta))}|u(y)|^{2}\, dy \leqslant \|u\|_{\delta, \lambda, s}^{2} \cdot (\gamma^{-2}(x, \delta)+M^{-\varepsilon}|\lambda|)^{-s} \operatorname{vol} B_{L}(x, \gamma(x, \delta)).$$

LEMMA 2.

$$\int\left(1+\frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-m}|u(y)|^{2}\, dy \leqslant C\|u\|_{\delta, \lambda, s}^{2} \cdot (\gamma^{-2}(x, \delta)+M^{-\varepsilon}|\lambda|)^{-s} \operatorname{vol} B_{L}(x, \gamma(x, \delta))$$

for m, C depending only on s.

Proof. Set

$$X=\int\left(1+\frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-m} \cdot\left\{\frac{(\gamma^{-2}(y, \delta)+M^{-\varepsilon}|\lambda|)^{s}}{\operatorname{vol} B_{L}(y, \gamma(y, \delta))} \int_{z \in B_L(y, \gamma(y, \delta))}|u(z)|^{2}\, dz\right\} dy.$$

The expression in curly brackets is at most $\|u\|^2_{\delta,\lambda,s}$, so (if $\bar{m}$ is big)

$$X \leqslant \|u\|^2_{\delta,\lambda,s} \int \left(1+\frac{\operatorname{dist}(x,y)}{\gamma(x,\delta)}\right)^{-\bar{m}} dy \leqslant C\,\|u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x,\gamma(x,\delta)).$$

On the other hand, $y \in B_L(z, c_1\gamma(z,\delta))$ for $c_1 \ll 1$ implies $\gamma(z,\delta) \sim \gamma(y,\delta)$, and therefore $z \in B_L(y,\gamma(y,\delta))$. Hence, switching the order of integration in the definition of X, we get

$$X \geqslant \int |u(z)|^2 \cdot \left\{ \int_{y \in B_L(z,c_1\gamma(z,\delta))} \left(1+\frac{\operatorname{dist}(x,y)}{\gamma(x,\delta)}\right)^{-\bar{m}} \frac{(\gamma^{-2}(y,\delta)+M^{-\varepsilon}|\lambda|)^s}{\operatorname{vol} B_L(y,\gamma(y,\delta))}\,dy \right\} dz.$$

In the inner integral, $\gamma(y,\delta) \sim \gamma(z,\delta)$, $\operatorname{vol} B_L(y,\gamma(y,\delta)) \sim \operatorname{vol} B_L(z,\gamma(z,\delta))$. Also

$$\begin{aligned} 1+\frac{\operatorname{dist}(x,y)}{\gamma(x,\delta)} &= \frac{\gamma(x,\delta)+\operatorname{dist}(x,y)}{\gamma(x,\delta)} \\ &\leqslant \frac{\gamma(x,\delta)+\operatorname{dist}(x,z)+\operatorname{dist}(y,z)}{\gamma(x,\delta)} \\ &\leqslant 1+\frac{\operatorname{dist}(x,z)}{\gamma(x,\delta)}+c_1\frac{\gamma(z,\delta)}{\gamma(x,\delta)} \\ &\leqslant C\left(1+\frac{\operatorname{dist}(x,z)}{\gamma(x,\delta)}\right) \end{aligned}$$

by (a) of Lemma 1. Putting these estimates into our lower bound for X gives

$$X \geqslant \int |u(z)|^2 \cdot c\left(1+\frac{\operatorname{dist}(x,z)}{\gamma(x,\delta)}\right)^{-\bar{m}} \frac{(\gamma^{-2}(z,\delta)+M^{-\varepsilon}|\lambda|)^s}{\operatorname{vol} B_L(z,\gamma(z,\delta))} \cdot \operatorname{vol} B_L(z,c_1\gamma(z,\delta))\,dz.$$

Comparing our upper and lower bounds for X and recalling that $\operatorname{vol} B_L(z,c_1\gamma(z,\delta))/\operatorname{vol} B_L(z,\gamma(z,\delta)) \geqslant c_2$, we conclude that

$$\int |u(z)|^2 \cdot \left(1+\frac{\operatorname{dist}(x,z)}{\gamma(x,\delta)}\right)^{-\bar{m}} (\gamma^{-2}(z,\delta)+M^{-\varepsilon}|\lambda|)^s\,dz \leqslant C\,\|u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x,\gamma(x,\delta)).$$

The corollary to Lemma 1 with $D = M^{-\varepsilon}|\lambda|$ gives

$$(\gamma^{-2}(z, \delta) + M^{-\varepsilon}|\lambda|)^s \geqslant c(\gamma^{-2}(x, \delta) + M^{-\varepsilon}|\lambda|)^s \cdot \left(1 + \frac{\operatorname{dist}(x, z)}{\gamma(x, \delta)}\right)^{-K|s|}.$$

Thus

$$\int |u(z)|^2 \left(1 + \frac{\operatorname{dist}(x, z)}{\gamma(x, \delta)}\right)^{-\bar{m} - |s| K} dz$$
$$\leqslant C \|u\|^2_{\delta, \lambda, s} \operatorname{vol} B_L(x, \gamma(x, \delta)) \cdot (\gamma^{-2}(x, \delta) + M^{-\varepsilon}|\lambda|)^{-s}.$$

The lemma is proved, with $m = \bar{m} + |s| K$. ▮

COROLLARY. *Suppose the straightening of $B_L(x, \gamma_0)$ is given by Φ: $\prod_{k=1}^n I_k^+ \to \mathbb{R}^n$, with $\gamma_0 = \gamma(x, \delta)$, $I_k = \{|x_k| < \delta_k\}$, $I_k^+ = \{|x_k| < \delta^{-\varepsilon}\delta_k\}$, $|(\Phi')^{\pm 1}| \leqslant C$, etc. Then*

$$\int_{\prod_k I_k^+} |u \circ \Phi(y)|^2 \cdot \left(1 + \sum_{k=1}^n \frac{|y_k|}{\delta_k}\right)^{-m} dy$$
$$\leqslant C \|u\|^2_{\delta, \lambda, s} \operatorname{vol} B_L(x, \gamma_0) \cdot (\gamma_0^{-2} + M^{-\varepsilon}|\lambda|)^{-s}$$

with C, m depending only on s.

Proof. Set $z = \Phi(y)$. This change of variables has a bounded Jacobian determinant, and

$$c\left(1 + \frac{\operatorname{dist}(x, z)}{\gamma_0}\right)^{\varepsilon} \leqslant \left(1 + \sum_{k=1}^n \frac{|y_k|}{\delta_k}\right) \leqslant C\left(1 + \frac{\operatorname{dist}(x, z)}{\gamma_0}\right)^K.$$

Hence the corollary follows from the preceding lemma. ▮

The L^∞-analogue of the preceding lemma is as follows:

LEMMA 3.

$$\sup_y \left\{|u(y)| \cdot \left(1 + \frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-m}\right\} \leqslant C \, |||u|||_{\delta, \lambda, s} \cdot (\gamma(x, \delta)^{-2} + M^{-\varepsilon}|\lambda|)^{-s/2}.$$

Proof.

$$|u(y)| \cdot \left(1 + \frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-m} \leqslant C \, |||u|||_{\delta, \lambda, s}$$
$$\cdot (\gamma(y, \delta)^{-2} + M^{-\varepsilon}|\lambda|)^{-s/2} \cdot \left(1 + \frac{\operatorname{dist}(x, y)}{\gamma(x, \delta)}\right)^{-m}$$
$$\leqslant C \, |||u|||_{\delta, \lambda, s} \cdot (\gamma(x, \delta)^{-2} + M^{-\varepsilon}|\lambda|)^{-s/2}$$

by the Corollary to Lemma 1 with $D = M^{-\varepsilon}|\lambda|$. ▮

LEMMA 4. *Let Q be a pseudodifferential operator of order zero with symbol $Q(x, \xi)$ supported in $\{|\xi| \sim M/\delta\}$. Then*

$$\|Qu\|_{\delta,\lambda,s} \leqslant C\, \|u\|_{\delta,\lambda,s} \tag{a}$$

$$|||Qu|||_{\delta,\lambda,s} \leqslant C\, |||u|||_{\delta,\lambda,s}. \tag{b}$$

Let $\sigma \in C_0^\infty(B_L(x, \gamma(x, \delta)))$. Suppose $|\nabla\sigma| \leqslant C/\delta$. Then

$$\|[\sigma, Q]u\|_{\delta,\lambda,s} \leqslant \frac{C}{M} \|u\|_{\delta,\lambda,s}. \tag{c}$$

Proof. $Qu(y) = \int_{\mathbb{R}^n} K(y, z)\, u(z)\, dz$ with $|K(y, z)| \leqslant C_{\bar{m}}(M/\delta)^n (1+|y-z|\cdot M/\delta)^{-\bar{m}}$ (any $\bar{m}$). To see this, write out the definition $Qu(y) = \int e^{i\xi\cdot(y-z)}\, Q(y, \xi)\, u(z)\, dz\, d\xi$, substitute $e^{i\xi\cdot(y-z)} = [(I-(M/\delta)^2 \Delta_\xi)(1+(M^2/\delta^2)\,|y-z|^2)^{-1}]^m e^{i\xi\cdot(y-z)}$, and integrate by parts repeatedly to obtain

$$Qu(y) = \int \Bigg[\int e^{i\xi\cdot(y-z)} \left(1+\frac{M^2}{\delta^2}|y-z|^2\right)^{-m} \times \left(I - \left(\frac{M}{\delta}\right)^2 \Delta_\xi\right)^m Q(y, \xi)\, d\xi \Bigg] u(z)\, dz,$$

proving the estimate on K. (This is an old and standard trick.) Thus for $y \in B_L(x, \gamma(x, \delta))$ we have

$$\begin{aligned} |Qu(y)|^2 &\leqslant C \int_{\mathbb{R}^n} |u(z)|^2 \cdot \left(\frac{M}{\delta}\right)^n \times \left(1+\frac{M}{\delta}|y-z|\right)^{-\bar{m}} dz \qquad \text{(by Cauchy–Schwartz)} \\ &\leqslant C \int_{\mathbb{R}^n} |u(z)|^2 \cdot \left(1+\frac{\operatorname{dist}(x, z)}{\gamma(x, \delta)}\right)^{-m} \cdot \left(\frac{M}{\delta}\right)^n \left(1+\frac{M}{\delta}|y-z|\right)^{-\bar{m}/2} dz \end{aligned} \tag{+}$$

if $\bar{m}$ is large, depending on m.

To see the last estimate, we work in straightened coordinates $y = \Phi(\bar{y})$, $z = \Phi(\bar{z})$, assuming $z \in \Phi(\prod_k I_k^+)$. We have $1 + (M/\delta)\,|y-z| \sim 1 + M \sum_k (|\bar{y}_k - \bar{z}_k|/\delta) \geqslant 1 + \sum_k (|\bar{y}_k - \bar{z}_k|/\delta_k) \geqslant c\cdot(1+\sum_k (|\bar{z}_k|/\delta_k))$ (since $|\bar{y}_k| \leqslant C\,\delta_k) \geqslant c'(1+\operatorname{dist}(x, z)/\gamma(x, \delta))^\varepsilon$. Hence $(1+\operatorname{dist}(x, z)/\gamma(x, \delta))^{-m} \leqslant C(1+(M/\delta)\,|y-z|)^{-m/\varepsilon}$. This takes care of the integrals over $z \in \Phi(\prod_k I_k^+)$ in (+). The region $z \notin \Phi(\prod_k I_k^+)$ is trivial, since $(1+(M/\delta)\,|y-z|)^{-\bar{m}/2} <$

$C\delta^{\varepsilon\bar{m}/2}$. Thus, $(+)$ is proved, for $y \in B_L(x, \gamma(x, \delta))$. Integrating $(+)$ over $B_L(x, \gamma(x, \delta))$, we have

$$\begin{aligned}
&\int_{B_L(x,\, \gamma(x,\delta))} |Qu(y)|^2\, dy \\
&\quad \leqslant C \int_{\mathbb{R}^n} |u(z)|^2 \cdot \left(1 + \frac{\operatorname{dist}(x, z)}{\gamma(x, \delta)}\right)^{-m} dz \\
&\quad \leqslant C' \, \|u\|^2_{\delta, \lambda, s} \operatorname{vol} B_L(x, \gamma(x, \delta)) \cdot (\gamma(x, \delta)^{-2} + M^{-\varepsilon} |\lambda|)^{-s},
\end{aligned}$$

where the last estimate follows from Lemma 2. Hence

$$\begin{aligned}
\|Qu\|^2_{\delta, \lambda, s} = \sup_x \Big\{ \int_{B_L(x, \gamma(x,\delta))} &|Qu(y)|^2\, dy / [\operatorname{vol} B_L(x, \gamma(x, \delta)) \\
&\cdot (\gamma(x, \delta)^{-2} + M^{-\varepsilon} |\lambda|)^{-s}] \Big\} \\
\leqslant C' \, &\|u\|^2_{\delta, \lambda, s}
\end{aligned}$$

which proves (a).

Similarly, taking $y = x$ in $(+)$, we see that

$$\begin{aligned}
|Qu(x)| &\leqslant C \sup_z \left\{ |u(z)| \cdot \left(1 + \frac{\operatorname{dist}(z, x)}{\gamma(x, \delta)}\right)^{-m/2} \right\} \\
&\leqslant C' \, |||u|||_{\delta, \lambda, s} \cdot (\gamma(x, \delta)^{-2} + M^{-\varepsilon} |\lambda|)^{-s/2}
\end{aligned}$$

(by Lemma 3), which is equivalent to saying that $|||Qu|||_{\delta, \lambda, s} \leqslant C' \, |||u|||_{\delta, \lambda, s}$. This proves (b).

To prove (c), we write

$$M[\sigma, Q]\, u(y) = \int M(\sigma(y) - \sigma(z))\, K(y, z)\, u(z)\, dz \equiv \int \tilde{K}(y, z)\, u(z)\, dz.$$

Since $|\nabla\sigma| \leqslant C/\sigma$, we get $M\, |\sigma(y) - \sigma(z)| \leqslant (M/\delta)\, |y - z| \leqslant 1 + (M/\delta)\, |y - z|$. Hence $\tilde{K}(y, z)$ satisfies the same estimate which we used for $K(y, z)$. Therefore, the proof of (a) yields (c) as well. ∎

5. PDE Lemmas

Let L be a self-adjoint second-order real subelliptic PDE on $|x| < 2$ with non-negative principal symbol. Let $\varkappa_\nu$, Y, Y' be smooth subunit vector fields.

Let $0<\eta<1/2$ be a small positive number.

LEMMA A. *If* $LU=F_0+\sum_{\nu\geqslant 1}\varkappa_\nu F_\nu+YY'F_*$ *in* $|y|<1$, *then*

$$|U(0)|\leqslant C\,\|F_0\|_{L^\infty}+\sum_{\nu\geqslant 1} C\,\|F_\nu\|_{L^\infty}+C\,|\ln\eta|\;\|F_*\|_{L^\infty}$$
$$+C\,\|U\|_{L^2}+C\eta^2\,\|YY'F_*\|_{L^\infty}.$$

LEMMA B. *If* $LU=F_0+\sum_{\nu\geqslant 1}\varkappa_\nu F_\nu$ *in* $|y|<1$, *then*

$$|YU(0)|\leqslant C\,\|F_0\|_{L^\infty}+C\,|\ln\eta|\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty}+C\,\|U\|_{L^2}$$
$$+C\eta\sum_{\nu\geqslant 1}\|\varkappa_\nu F_\nu\|_{L^\infty}.$$

LEMMA C. *If* $LU=F_0$ *in* $|y|<1$, *then*

$$|YY'U(0)|\leqslant C\,|\ln\eta|\;\|F_0\|_{L^\infty}+C\,\|U\|_{L^2}+C\eta\,\|U\|_{C^3}.$$

In Lemmas A, B, C and Lemmas D, E below, the norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{C^3}$, etc., refer to the region $|y|<1$.

We need also two refinements of Lemma A.

LEMMA D. *Let* $0<a<1/2$ *be a small positive number. If* $LU=F_0+\sum_\nu\varkappa_\nu F_\nu+YY'F_*$ *in* $|y|<1$, *then*

$$|U(0)|\leqslant a\,\|F_0\|_{L^\infty}+C\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty}+C\,|\ln\eta|\;\|F_*\|_{L^\infty}$$
$$+C(a)\,\|U\|_{L^2}+C\eta^2\,\|YY'F_*\|.$$

LEMMA E. *Suppose* λ *is a large complex number* ($|\lambda|\geqslant B^2$, *large constant to be fixed later*), *and suppose* $|\mathrm{Im}\,\lambda|\geqslant c\,|\lambda|$.

If $(L-\lambda)U=F_0+\sum_{\nu\geqslant 1}\varkappa_\nu F_\nu$ *in* $|y|<1$, *then*

$$|U(0)|\leqslant\frac{C}{|\lambda|}\,\|F_0\|_{L^\infty}+\frac{C}{|\lambda|^{1/2}}\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty}+\frac{C}{B}\,\|U\|_{L^\infty}.$$

Remark. In our application of Lemmas A–E, we will take η to be so small that the terms $\eta^2\,\|YY'F_+\|_{L^\infty}$ in Lemmas A and D, $C\eta\sum_{\nu\geqslant 1}\|\varkappa_\nu F_\nu\|_{L^\infty}$ in Lemma B, and $C\eta\,\|U\|_{C^3}$ in Lemma C will become negligibly small.

Proof of Lemmas A–E. For Lemmas A, B, C we use the known form of the fundamental solution of L (see [FS]). Lemmas D and E follow from A, B, C by appropriate rescaling. Details are as follows.

The equation $Lu = F_0 + \sum_{v \geqslant 1} \varkappa_v F_v$ in $|x| < 1/2$ ($\varkappa_v$ smooth, subunit in $|x| < 1$) has a particular solution

$$u_1(x) = \int_{|y|<1} K_0(x, y)\, F_0(y)\, dy$$
$$+ \sum_{v \geqslant 1} \int_{|y|<1} K_v(x, y)\, F_v(y)\, dy \qquad \text{in} \quad |x| < 1/2 \tag{1}$$

with

$$|K_0(x, y)| \leqslant \frac{C \operatorname{dist}^2(x, y)}{\operatorname{vol}(x, y)}, \qquad |K_v(x, y)| \leqslant \frac{C \operatorname{dist}(x, y)}{\operatorname{vol}(x, y)}$$

$$|YK_0(x, y)| \leqslant \frac{C \operatorname{dist}(x, y)}{\operatorname{vol}(x, y)}, \qquad |YK_v(x, y)| \leqslant \frac{C}{\operatorname{vol}(x, y)}$$

$$|YY'K_0(x, y)| \leqslant \frac{C}{\operatorname{vol}(x, y)}.$$

Here $\operatorname{vol}(x, y) = \operatorname{vol} B_L(x, \operatorname{dist}(x, y))$; and Y, Y' are smooth subunit vector fields acting on the x-variable in $K_v(x, y)$. These results are contained in [FS].

Fix a cutoff function $\sigma \in C_0^\infty(B_L(0, \eta))$ with $\sigma = 1$ in $B_L(0, (2/3)\eta)$, $|Y\sigma| \leqslant C\eta^{-1}$, $|YY'\sigma| \leqslant C\eta^{-2}$, and $\eta \ll 1$ to be picked.

Since our PDE may be rewritten as

$$Lu = \left\{F_0 + \sum_{v \geqslant 1} \varkappa_v(\sigma F_v)\right\} + \sum_{v \geqslant 1} \varkappa_v\{(1-\sigma) F_v\} \equiv \tilde{F}_0 + \sum_v \tilde{F}_v,$$

we may apply (1) to $\tilde{F}_0$, $\tilde{F}_v$ to obtain a second solution, namely

$$u_2(x) = \int_{|y|<1} K_0(x, y) \left\{F_0(y) + \sigma(y) \sum_v \varkappa_v F_v(y) + \sum_v (\varkappa_v \sigma) \cdot F_v(y)\right\} dy$$
$$+ \sum_{v \geqslant 1} \int_{|y|<1} K_v(x, y)\{(1 - \sigma(y))\, F_v(y)\}\, dy \qquad \text{in} \quad |y| < 1/2. \tag{2}$$

In Eq. (2) there is no trouble differentiating under the integral sign to obtain

$$Yu_2(x) = \int_{|y|<1} (YK_0)(x, y) \Big\{ F_0(y) + \sigma(y) \times \sum_\nu \varkappa_\nu F_\nu(y) + \sum_\nu F_\nu(y) \cdot \varkappa_\nu \sigma(y) \Big\} \, dy + \sum_{\nu \geq 1} \int_{|y|<1} (YK_\nu)(x, y) \times \{(1 - \sigma(y)) F_\nu(y)\} \, dy \quad \text{for} \quad x \in B_L(0, \tfrac{1}{2}\eta). \tag{3}$$

Next we seek a particular solution to the equation

$$Lu = YY'F_* \quad \text{in} \quad |x| < \tfrac{1}{2}. \tag{4}$$

We fix a cutoff $\theta \in C_0^\infty(|x| < 1)$ equal to 1 in $\{|x| < 1/2\}$ and write

$$u_3(x) = \int_{|y|<1} K_0(x, y) \, \theta(y) \, YY'F_*(y) \, dy = \int K_0(x, y) \, \sigma(y) \, YY'F_*(y) \, dy + \int [(YY')^* \{K_0(x, y) \, \theta(y)(1 - \sigma(y))\}] \cdot F_*(y) \, dy \quad \text{in} \quad B_L(0, \tfrac{1}{2}\eta), \tag{5}$$

where $(YY')^*$ acts on the y-variable in the expansion in curly brackets.

Next we want to understand $YY'u$ for a particular solution of $Lu = F_0$. Let $\sigma_1 \in C_0^\infty(B_L(0, 3\eta/8))$ with $\sigma_1 = 1$ in $B_L(0, \eta/4)$, $\sigma_1 \geq 0$ everywhere, $|Y\sigma_1| \leq C\eta^{-1}$, $|YY'\sigma_1| \leq C\eta^{-2}$. We use the particular solution (1) with $F_\nu = 0$ $(u \geq 1)$ and write

$$\int \sigma_1 \, YY'u_1 \, dx = \int \sigma_1(x) \, YY'K_0(x, y)\{(1 - \sigma(y)) \, F_0(y)\} \, dx \, dy + \int (Y^*\sigma_1(x)) \, Y'K_0(x, y) \, \sigma(y) \times F_0(y) \, dx \, dy \quad (\text{when } F_\nu = 0 \text{ for } \nu \geq 1). \tag{6}$$

Now we shall make obvious estimates for the integrals in (1), (2), (3), (5), (6) simply by putting absolute values inside the integrals and recalling our estimates for $|K_\nu(x, y)|$, $|YK_\nu(x, y)|$, $|Y\sigma(y)|$, etc. We obtain the following results.

$$|u_1(x)| \leqslant C \|F_0\|_{L^\infty(|y|<1)} + C \sum_{v \geqslant 1} \|F_v\|_{L^\infty(|y|<1)} \qquad \text{for} \quad |x| < 1/2 \tag{7}$$

$$|u_2(x)| \leqslant C \|F_0\|_{L^\infty(|y|<1)} + C \sum_{v \geqslant 1} \|F_v\|_{L^\infty(|y|<1)}$$

$$+ C\eta^2 \sum_{v \geqslant 1} \|\varkappa_v F_v\|_{L^\infty(|y|<1)} \qquad \text{for} \quad |x| < 1/2 \tag{8}$$

$$\|Yu_2(x)\| \leqslant C \|F_0\|_{L^\infty(|y|<1)} + C |\ln \eta| \sum_{v \geqslant 1} \|F_v\|_{L^\infty(|y|<1)}$$

$$+ C\eta \sum_{v \geqslant 1} \|\varkappa_v F_v\|_{L^\infty(|y|<1)} \qquad \text{for} \quad x \in B_L\left(0, \frac{1}{2}\eta\right) \tag{9}$$

$$|u_3(x)| \leqslant C |\ln \eta| \; \|F_*\|_{L^\infty(|y|<1)}$$
$$+ C\eta^2 \|YY'F_*\|_{L^\infty(|y|<1)} \qquad \text{for} \quad x \in B_L(0, \tfrac{1}{2}\eta) \tag{10}$$

$$\left| \int \sigma_1 YY'u_1 \, dx \right| \leqslant C |\ln \eta| \; \|F_0\|_{L^\infty(|y|<1)}$$
$$\cdot \operatorname{vol} B_L(0, \eta) \qquad \text{when} \quad F_v \equiv 0 (v < 1). \tag{11}$$

Next suppose $LU = F_0 + \sum_{v \geqslant 1} \varkappa_v F_v$. Then $L(U - u_1) = L(U - u_2) = 0$ in $|x| < 1/2$. Since L is subelliptic, we have

$$\begin{aligned} \|U - u_1\|_{C^3(|x|<1/4)} &\leqslant C \|U - u_1\|_{L^2(|x|<1/2)} \\ &\leqslant C \|F_0\|_{L^\infty(|y|<1)} \\ &\quad + \sum_{v \geqslant 1} C \|F_v\|_{L^\infty(|y|<1)} + C \|U\|_{L^2(|x|<1/2)} \end{aligned} \tag{12}$$

by virtue of (7); and similarly

$$\begin{aligned} \|U - u_2\|_{C^3(|x|<1/4)} &\leqslant C \|U - u_2\|_{L^2(|x|<1/2)} \\ &\leqslant C \|F_0\|_{L^\infty(|y|<1)} + C \sum_{v \geqslant 1} \|F_v\|_{L^\infty(|y|<1)} \\ &\quad + C\eta^2 \sum_{v \geqslant 1} \|\varkappa_v F_v\|_{L^\infty(|y|<1)} + C \|U\|_{L^2(|y|<1)}. \end{aligned} \tag{13}$$

By (9) and (13) we have

$$
\begin{aligned}
|YU(0)| &\leqslant |Yu_2(0)| + C\,\|U-u_2\|_{C^3(|x|<1/4)}\\
&\leqslant C\,\|F_0\|_{L^\infty(|y|<1)} + C\,|\ln\eta| \sum_{\nu\geqslant 1} \|F_\nu\|_{L^\infty(|y|<1)}\\
&\quad + C\eta \sum_{\nu\geqslant 1} \|\varkappa_\nu F_\nu\|_{L^\infty(|y|<1)} + C\,\|U\|_{L^2(|y|<1)}.
\end{aligned}
\tag{14}
$$

Similarly, by (11) and (12) with $F_\nu \equiv 0$ $(\nu \geqslant 1)$, we have

$$
\begin{aligned}
\left|\int \sigma_1\, YY'U\,dx\right| &\leqslant \left|\int \sigma_1\, YY'u_1\,dx\right|\\
&\quad + C \operatorname{vol} B_L(0,\eta)\cdot \|U-u_1\|_{C^3(|x|<1/4)}\\
&\leqslant \{C\,|\ln\eta|\,\|F_0\|_{L^\infty(|y|<1)} + C\,\|U\|_{L^2(|y|<1)}\}\\
&\quad \times \operatorname{vol} B_L(0,\eta) \qquad \text{if} \quad F_\nu\equiv 0 (\nu\geqslant 1).
\end{aligned}
\tag{15}
$$

Moreover, with $V=\int_{\mathbb{R}^n} \sigma_1\,dx \sim \operatorname{vol} B_L(0,\eta)$ we have

$$
|YY'U(0)| \leqslant \left|\frac{1}{V}\int \sigma_1 YY'U\,dx\right| + C\eta\,\|U\|_{C^3(|y|<1)},
$$

since $x\in \operatorname{supp}\sigma_1$ implies

$$
\begin{aligned}
|YY'U(x) - YY'U(0)| &\leqslant \max_{|y|<1} |\nabla YY'U(y)|\cdot \operatorname{diam}(\operatorname{supp}\sigma_1)\\
&\leqslant C\,\|U\|_{C^3}\cdot\eta.
\end{aligned}
$$

Therefore, (15) implies

$$
\begin{aligned}
|YY'U(0)| &\leqslant C\,|\ln\eta|\,\|F_0\|_{L^\infty(|y|<1)} + C\,\|U\|_{L^2(|y|<1)}\\
&\quad + C\eta\,\|U\|_{C^3(|y|<1)} \qquad \text{provided} \quad F_\nu\equiv 0 \text{ for } \nu\geqslant 1.
\end{aligned}
\tag{16}
$$

Similarly, suppose

$$
LU = F_0 + \sum_{\nu\geqslant 1} z_\nu F_\nu + YY'F_*
$$

in $|x|<1$. Then

$$
L(U-u_1-u_3)=0
$$

in $|x| < 1/2$, so that

$$\|U - u_1 - u_3\|_{C^3(|y| < 1/4)} \leqslant C \|U - u_1 - u_3\|_{L^2(|y| < 1/2)} \leqslant C \|U\|_{L^2(|y| < 1)}$$

$$+ C \|F_0\|_{L^\infty(|y| < 1)} + C \sum_{\nu \geqslant 1} \|F_\nu\|_{L^\infty(|y| < 1)}$$

$$+ C |\ln \eta| \|F_*\|_{L^\infty(|y| < 1)} + C\eta^2 \|YY'F_*\|_{L^\infty(|y| < 1)}$$

by virtue of (7) and (10). Hence

$$|U(0)| \leqslant |u_1(0)| + |u_3(0)| + \|U - u_1 - u_3\|_{C^3(|y| < 1/4)} \leqslant C \|U\|_{L^2(|y| < 1)}$$

$$+ C \|F_0\|_{L^\infty(|y| < 1)} + C \sum_{\nu \geqslant 1} \|F_\nu\|_{L^\infty(|y| < 1)}$$

$$+ C |\ln \eta| \|F_*\|_{L^\infty(|y| < 1)} + C\eta^2 \|YY'F_*\|_{L\infty(|y| < 1)} \tag{17}$$

by another application of (7) and (10).

Estimates (14), (16), (17) complete the proof of Lemmas A, B, C. ∎

Proof of Lemma D. If $LU = F_0 + \sum_{\nu \geqslant 1} \varkappa_\nu F_\nu + YY'F_*$ in $|y| < 1$, then

$$(\gamma^2 L) U = (\gamma^2 F_0) + \sum_{\nu \geqslant 1} (\gamma \varkappa_\nu)(\gamma F_\nu) + (\gamma Y)(\gamma Y')F_*$$

in $B_L(0, \gamma)$, $0 < \gamma \ll 1$.

Changing to rescaled, straightened coordinates (i.e., composing with the map $\tilde{\Phi}$ in the section on Geometry) and invoking Lemma A, we obtain the estimate

$$|U(0)| \leqslant C \|\gamma^2 F_0\|_{L^\infty(B_L(0,\gamma))} + C \sum_{\nu \geqslant 1} \|\gamma F_\nu\|_{L^\infty(B_L(0,\gamma))}$$

$$+ C |\ln \eta| \|F_*\|_{L^\infty(B_L(0,\gamma))}$$

$$+ \frac{C}{\sqrt{\operatorname{vol} B_L(0, \gamma)}} \|U\|_{L^2(B_L(0,\gamma))} + C\eta^2\gamma^2 \|YY'F_*\|_{L^\infty(B_L(0,\gamma))}.$$

Taking γ equal to a small constant times $\sqrt{a}$ and dominating L^∞ and L^2 norms over $B_L(0, \gamma)$ by norms over $\{|y| < 1\}$, we obtain Lemma D with $C(a) = C/\sqrt{\operatorname{vol} B_L(0, \gamma)}$. ∎

Proof of Lemma E. Suppose first that $\operatorname{supp} U \subset \{|y| < 1/2\}$. We shall first prove an L^2-estimate for U localized to a ball $B_L(x_0, |\lambda|^{-1/2})$, then we can invoke Lemma A on that ball.

Let $M \gg 1$ be a constant to be picked later and let x_0 belong to supp U. We introduce cutoff functions σ, $\sigma_1 \in C_0^\infty(B_L(x_0, M|\lambda|^{-1/2}))$ with the following properties: $\sigma_1 = 1$ on supp σ, $\sigma = 1$ on $B_L(x_0, (1/2)M\,|\lambda|^{-1/2})$, $|Y\sigma|$, $|Y\sigma_1| \leqslant C\,|\lambda|^{1/2}/M$ for subunit tangent vectors Y.

Let Y be a subunit tangent vector field, not assumed to be smooth. For w supported in $\{|y| < 1/2\}$ we have the L^2-estimates

$$\|Yw\|^2 \leqslant \langle Lw, w\rangle = \operatorname{Re}\langle Lw, w\rangle \leqslant \operatorname{Re}\langle (L-\lambda)w, w\rangle + |\lambda|\ \|w\|^2.$$

(The first inequality is an elementary integration by parts.)

$$|\operatorname{Im}\langle (L-\lambda)w, w\rangle| = |\operatorname{Im}\langle \lambda w, w\rangle| = |\operatorname{Im}\lambda|\ \|w\|^2 \geqslant c\,|\lambda|\ \|w\|^2.$$

Together, these estimates imply

$$|\lambda|\ \|w\|^2 + \|Yw\|^2 \leqslant C\,|\langle (L-\lambda)w, w\rangle| \qquad \text{for} \quad \operatorname{supp} w \subset \{|y| < 1/2\}.$$

We apply this to $w = \sigma u$. Note that

$$Yw = Y(\sigma u) = \sigma(Yu) + u(Y\sigma) \qquad \text{and} \qquad \|u(Y\sigma)\|^2 \leqslant \frac{C\,|\lambda|}{M^2}\,\|\sigma_1 u\|^2.$$

Hence we obtain

$$\begin{aligned} &|\lambda|\ \|\sigma u\|^2 + \|\sigma Y u\|^2 \\ &\quad \leqslant \frac{C\,|\lambda|}{M^2}\,\|\sigma_1 u\|^2 + C\,|\langle (L-\lambda)(\sigma u), (\sigma u)\rangle| \\ &\quad \leqslant \frac{C\,|\lambda|}{M^2}\,\|\sigma_1 u\|^2 + C\,|\langle \sigma(L-\lambda)u, \sigma u\rangle| \\ &\qquad + C\,|\langle [L,\sigma]u, \sigma u\rangle|. \end{aligned} \tag{e1}$$

Now $[L,\sigma] = (C\lambda^{1/2}/M)\,\sigma_1 \mathscr{W} + (b\,|\lambda|/M^2)\sigma_1$ with $\mathscr{W}$ subunit and $|b| \leqslant C$. (This is easily verified by viewing $[(M^2/|\lambda|)L, \sigma]$ in rescaled, straightened coordinates on $B_L(x_0, M/|\lambda|^{1/2})$.)

Hence

$$\begin{aligned} |\langle [L,\sigma]u, \sigma u\rangle| &\leqslant \frac{C\,|\lambda|^{1/2}}{M}\,|\langle \sigma_1 \mathscr{W} u, \sigma u\rangle| + \frac{C\,|\lambda|}{M^2}\,|\langle b\sigma_1 u, \sigma u\rangle| \\ &\leqslant \frac{C}{M}\,\|\sigma_1 \mathscr{W} u\|^2 + \frac{C\,|\lambda|}{M}\,\|\sigma u\|^2 + \frac{C\,|\lambda|}{M^2}\,\|\sigma_1 u\|^2 \\ &\leqslant \frac{C}{M}\,\|\sigma_1 \mathscr{W} u\|^2 + \frac{C\,|\lambda|}{M}\,\|\sigma_1 u\|^2. \end{aligned}$$

Putting this back into (e1), we see that

$$\{|\lambda|\ \|\sigma u\|^2+\|\sigma Yu\|^2\}\leqslant\frac{C}{M}\{|\lambda|\ \|\sigma_1 u\|^2+\|\sigma_1\mathscr{W}u\|^2\}$$
$$+C|\langle\sigma(L-\lambda)u,\sigma u\rangle|. \tag{e2}$$

Now suppose $(L-\lambda)u=f_0+\sum_\nu \varkappa_\nu f_\nu$ with $\varkappa_\nu$ subunit, smooth. Then

$$|\langle\sigma(L-\lambda)u,\sigma u\rangle|\leqslant|\langle\sigma f_0,\sigma u\rangle|+\sum_\nu|\langle\sigma\varkappa_\nu f_\nu,\sigma u\rangle|$$
$$\leqslant\frac{CM}{|\lambda|}\|\sigma f_0\|^2+\frac{C|\lambda|}{M}\|\sigma u\|^2+C\sum_\nu|\langle(\varkappa_\nu\sigma)f_\nu,\sigma u\rangle|$$
$$+C\sum_\nu|\langle\sigma f_\nu,[\varkappa_\nu^*,\sigma]u\rangle|+\sum_\nu|\langle\sigma f_\nu,\sigma\varkappa_\nu^* u\rangle|. \tag{e3}$$

We have $\varkappa_\nu^*=-\varkappa_\nu+b$ with $|b|\leqslant C$, so

$$|\langle\sigma f_\nu,\sigma\varkappa_\nu^* u\rangle|\leqslant|\langle\sigma f_\nu,\sigma\varkappa_\nu u\rangle|+|\langle\sigma f_\nu,\sigma bu\rangle|$$
$$\leqslant CM\|\sigma f_\nu\|^2+\frac{C}{M}\|\sigma\varkappa_\nu u\|^2+\frac{C}{M}\|\sigma u\|^2. \tag{e4}$$

Also, $|\varkappa_\nu\sigma|\leqslant C|\lambda|^{1/2}/M$, $|[\varkappa_\nu^*,\sigma]|\leqslant C|\lambda|^{1/2}/M$ (the left-hand side is a function of x), so

$$|\langle(\varkappa_\nu\sigma)f_\nu,\sigma u\rangle|+|\langle\sigma f_\nu,[\varkappa_\nu^*,\sigma]u\rangle|\leqslant\frac{C|\lambda|^{1/2}}{M}\langle|\sigma_1 f_\nu|,|\sigma_1 u|\rangle$$
$$\leqslant C\|\sigma_1 f_\nu\|^2+\frac{C|\lambda|}{M}\|\sigma_1 u\|^2.$$

Putting this and (e4) into (e3), we obtain

$$|\langle\sigma(L-\lambda)u,\sigma u\rangle|\leqslant\frac{CM}{|\lambda|}\|\sigma_1 f_0\|^2+CM\sum_{\nu\geqslant 1}\|\sigma_1 f_\nu\|^2$$
$$+\left\{\frac{C|\lambda|}{M}\|\sigma_1 u\|^2+\sum_\nu\frac{C}{M}\|\sigma_1\varkappa_\nu u\|^2\right\}.$$

Putting this into (e2), and taking $\mathscr{Y}=\mathscr{W}$ or one of the $\varkappa_k$, whichever has the largest $\|\sigma_1\mathscr{Y}u\|$, we obtain

$$\{|\lambda|\ \|\sigma u\|^2+\|\sigma Yu\|^2\}\leqslant\frac{C}{M}\{|\lambda|\ \|\sigma_1 u\|^2+\|\sigma_1\mathscr{Y}u\|^2\}$$
$$+\frac{CM}{|\lambda|}\|\sigma_1 f_0\|^2+CM\sum_{\nu\geqslant 1}\|\sigma_1 f_\nu\|^2. \tag{e5}$$

This holds for arbitrary subunit Y and some subunit $\mathscr{Y}$. Recall that $\sigma = 1$ on $B_L(x_0, M/2\,|\lambda|^{1/2})$, while σ_1 is supported in $B_L(x_0, M/|\lambda|^{1/2})$. Assuming $f_0, f_\nu \in L^\infty$, we see from (e5) that

$$\begin{aligned}
&\left\{|\lambda| \int_{B_L(x_0, M/2\,|\lambda|^{1/2})} |u(y)|^2\,dy + \int_{B_L(x_0, M/2\,|\lambda|^{1/2})} |Yu(y)|^2\,dy\right\} \\
&\quad \leqslant \frac{C}{M}\left\{|\lambda| \int_{B_L(x_0, M/|\lambda|^{1/2})} |u(y)|^2\,dy + \int_{B_L(x_0, M/|\lambda|^{1/2})} |\mathscr{Y}u(y)|^2\,dy\right\} \\
&\qquad + CM \operatorname{vol} B_L\left(x_0, \frac{M}{|\lambda|^{1/2}}\right) \cdot \left\{\frac{\|f_0\|^2_{L^\infty}}{|\lambda|} + \sum_{\nu \geqslant 1} \|f_\nu\|^2_{L^\infty}\right\}.
\end{aligned} \tag{e6}$$

Now we are led to define

$$\Omega = \sup_{\substack{x_0 \in \mathbb{R}^m \\ Y\,\text{subunit}}} \left[\frac{|\lambda| \int_{B_L(x_0, M/2|\lambda|^{1/2})} |u(y)|^2\,dy + \int_{B_L(x_0, M/2|\lambda|^{1/2})} |Yu(y)|^2 dy\}}{\operatorname{vol} B_L(x_0, M/|\lambda|^{1/2})}\right],$$

and Ω^* analogous to Ω with the integrals over $B_L(x_0, M/2\,|\lambda|^{1/2})$ replaced by integrals over $B_L(x_0, M/|\lambda|^{1/2})$. Estimate (e6) asserts that

$$\Omega \leqslant \frac{C}{M}\,\Omega^* + CM\left\{\frac{\|f_0\|^2_{L^\infty}}{|\lambda|} + \sum_{\nu \geqslant 1} \|f_\nu\|^2_{L^\infty}\right\}. \tag{e7}$$

On the other hand, $B_L(x_0, M/|\lambda|^{1/2})$ can be covered by a family of smaller balls $B_L(x_j, M/2\,|\lambda|^{1/2})$, $j = 1, ..., j_{\max}$, with $j_{\max}$ bounded independently of M. Hence $\Omega^* \leqslant C\Omega$ with C independent of M. Therefore, we may once and for all pick M equal to a large constant so that the term $(C/M)\Omega^*$ in (e7) can be absorbed into the left-hand side. Consequently,

$$\Omega \leqslant CM\left\{\frac{\|f_0\|^2_{L^\infty}}{|\lambda|} + \sum_{\nu \geqslant 1} \|f_\nu\|^2_{L^\infty}\right\}.$$

In particular, this means that

$$\begin{aligned}
&\frac{1}{\operatorname{vol} B(x_0, C\,|\lambda|^{-1/2})}\,|\lambda| \int_{B(x_0, C|\lambda|^{-1/2})} |u(y)|^2\,dy \\
&\quad \leqslant C' \frac{\|f_0\|^2_{L^\infty}}{|\lambda|} + C' \sum_{\nu \geqslant 1} \|f_\nu\|^2_{L^\infty}.
\end{aligned} \tag{e8}$$

In (e8) and from now on in the proof of Lemma E, we stop keeping track of M dependence, and just regard M as another constant C.

We have proved (e8) for u supported in $\{|y| < 1/2\}$ and satisfying $Lu = f_0 + \sum_\nu \varkappa_\nu f_\nu$. Now we rewrite our equation for u as

$$(|\lambda|^{-1} L)u = (|\lambda|^{-1} f_0) + \sum_{\nu \geq 1} (|\lambda|^{-1/2} \varkappa_\nu)(|\lambda|^{-1/2} f_\nu)$$

in $B_L(x_0, |\lambda|^{-1/2})$.

Viewing this equation in rescaled, straightened coordinates (i.e., composing with the map $\tilde{\Phi}$ in the section on Geometry), and applying Lemma A with $F_* = 0$, we get the estimate

$$|u(x_0)| \leq \frac{C}{|\lambda|} \|f_0\|_{L^\infty} + \frac{C}{|\lambda|^{1/2}} \sum_{\nu \geq 1} \|f_\nu\| + \frac{C \, \|u\|_{L^2(B_L(x_0, |\lambda|^{-1/2}))}}{\sqrt{\mathrm{vol}\, B_L(x_0, |\lambda|^{-1/2})}}.$$

The crucial point is that our localized L^2-estimate (e8) shows that the last term on the right-hand side is dominated by the remaining terms on the right and may therefore be omitted.

Thus, we have proved the estimate

$$\|u\|_{L^\infty} \leq \frac{C}{|\lambda|} \|f_0\|_{L^\infty} + \sum_\nu \frac{C}{|\lambda|^{1/2}} \|f_\nu\|_{L^\infty} \tag{e9}$$

for u supported in $\{|y| < 1/2\}$ and satisfying $Lu = f_0 + \sum \varkappa_\nu f_\nu$, $\varkappa_\nu$ smooth, subunit. It remains to remove the restriction on the support of u.

Thus, suppose $(L - \lambda) U = F_0 + \sum_{\nu \geq 1} \varkappa_\nu F_\nu$ as in the hypothesis of Lemma E.

Let $\theta \in C_0^\infty(|y| < 1/2)$ with $\theta(0) = 1$. We have $L(\theta U) = \theta L U + \mathscr{W} U + (L\theta) U$ with $\mathscr{W}$ smooth and subunit. Hence

$$\begin{aligned} L(\theta U) &= \theta F_0 + \sum_{\nu \geq 1} \theta \varkappa_\nu F_\nu + \mathscr{W} U + (L\theta) U \\ &= \Big\{ \theta F_0 - \sum_{\nu \geq 1} (\varkappa_\nu \theta) \cdot F_\nu + (L\theta) \cdot U \Big\} + \sum_{\nu \geq 1} \varkappa_\nu (\theta F_\nu) + \mathscr{W} U \\ &\equiv f_0 + \sum_{\nu \geq 1} \varkappa_\nu (\theta F_\nu) + \mathscr{W} U. \end{aligned}$$

The last term on the right is to be regarded as merely an additional $\varkappa_\nu f_\nu$. Since θU is supported in $\{|y| \leq 1/2\}$, estimate (e9) applies, telling us that

$$\begin{aligned} |U(0)| \leq \|\theta U\|_{L^\infty} &\leq \frac{C}{|\lambda|} \|f_0\|_{L^\infty} + \sum_{\nu \geq 1} \frac{C}{|\lambda|^{1/2}} \|\theta F_\nu\|_{L^\infty} \\ &\quad + \frac{C}{|\lambda|^{1/2}} \|U\|_{L^\infty}, \end{aligned}$$

i.e.,

$$|U(0)| \leqslant \frac{C}{\lambda} \|F_0\|_{L^\infty} + \sum_{\nu \geqslant 1} \frac{C}{|\lambda|^{1/2}} \|F_\nu\|_{L^\infty} + \frac{C}{|\lambda|^{1/2}} \|U\|_{L^\infty},$$

the L^∞-norms taken over $\{|y| < 1\}$. Since $|\lambda| > B^2$, this implies the conclusion of Lemma E, completing the proof. ∎

The proof of Lemma E is really our whole proof for $(A-\lambda)^{-1}$ in miniature. Lemmas A–E refer to solutions of PDE in the unit cube. We need also their analogues for solutions of PDE in a small non-Euclidean ball $B_L(x_0, \gamma)$. Specifically, we have the following results:

LEMMA B′. *If $LU = F_0 + \sum_{\nu \geqslant 1} \varkappa_\nu F_\nu$ in $\mathscr{B} = B_L(x_0, \gamma)$ with $\varkappa_\nu$ smooth, subunit, then for subunit tangent vectors Y we have*

$$|YU(x_0)| \leqslant C\gamma \|F_0\|_{L^\infty(\mathscr{B})} + C |\ln \eta| \sum_{\nu \geqslant 1} \|F_\nu\|_{L^\infty(\mathscr{B})} + \frac{C \|U\|_{L^2(\mathscr{B})}}{\gamma \sqrt{\operatorname{vol} \mathscr{B}}} + C\eta\gamma \sum_{\nu \geqslant 1} \|\varkappa_\nu F_\nu\|_{L^\infty(\mathscr{B})}.$$

LEMMA C′. *If $LU = F_0$ in $\mathscr{B} = B_L(x_0, \gamma)$ then for smooth subunit vector fields Y, Y' we have*

$$|YY'U(x_0)| \leqslant C |\ln \eta| \|F_0\|_{L^\infty(\mathscr{B})} + \frac{C \|U\|_{L^2(\mathscr{B})}}{\gamma^2 \sqrt{\operatorname{vol} \mathscr{B}}} + \frac{C\eta}{\gamma^m} \|U\|_{C^3(\mathscr{B})}$$

for some fixed m depending only on L.

LEMMA D′. *If $0 < a < 1/2$ and $LU = F_0 + \sum_{\nu \geqslant 1} \varkappa_\nu F_\nu$ in $\mathscr{B} = B_L(x_0, \gamma)$ with $\varkappa_\nu$ smooth, subunit, then we have*

$$|U(x_0)| \leqslant a\gamma^2 \|F_0\|_{L^\infty(\mathscr{B})} + C\gamma \sum_{\nu \geqslant 1} \|F_\nu\|_{L^\infty(\mathscr{B})} + \frac{C(a) \|U\|_{L^2(\mathscr{B})}}{\sqrt{\operatorname{vol} \mathscr{B}}}.$$

LEMMA E′. *With $B \gg 1$ a large constant to be picked later, suppose $|\operatorname{Im} \lambda| \geqslant c |\lambda|$ and $|\lambda| \geqslant B^2 \gamma^{-2}$. If $(L-\lambda)U = F_0 + \sum_{\nu \geqslant 1} \varkappa_\nu F_\nu$ in $\mathscr{B} = B_L(x_0, \gamma)$ with $\varkappa_\nu$ smooth, subunit, then we have*

$$|U(x_0)| \leqslant \frac{C}{|\lambda|} \|F_0\|_{L^\infty(\mathscr{B})} + \frac{C}{|\lambda|^{1/2}} \sum_{\nu \geqslant 1} \|F_\nu\|_{L^\infty(\mathscr{B})} + \frac{C}{B} \|U\|_{L^\infty(\mathscr{B})}.$$

We have not stated an analogue for Lemma A simply because we will not need it.

Lemmas B′–E′ are easily deduced from Lemmas B–E by passing to rescaled, straightened coordinates, i.e., composing with the map $\tilde{\Phi}$ defined in the section on Geometry. We just set $\tilde{U}=U\circ\tilde{\Phi}$, $\tilde{F}_0=\gamma^2F_0\circ\tilde{\Phi}$, $\tilde{F}_\nu=\gamma F_\nu\circ\tilde{\Phi}$, and define $\tilde{\varkappa}_\nu$, $\tilde{Y}$, $\tilde{Y}'$, $\tilde{L}$ to be the pullbacks of $\gamma\varkappa_\nu$, γY, $\gamma Y'$, $\gamma^2 L$ by $\tilde{\Phi}$. Then $\tilde{\varkappa}_\nu$, $\tilde{Y}$, $\tilde{Y}'$, are subunit with respect to $\tilde{L}$ on the unit cube, $\tilde{L}$ is subelliptic, and the equation $LU=F_0+\sum_{\nu\geqslant 1}\varkappa_\nu F_\nu$ on $B_L(x_0,\gamma)$ goes over to $\tilde{L}\tilde{U}=\tilde{F}_0+\sum_{\nu\geqslant 1}\tilde{\varkappa}_\nu\tilde{F}_\nu$ on the unit cube. We note that the map $\tilde{\Phi}$ has C^3-norm bounded by some fixed negative power of γ, and the Jacobian determinant $\det\tilde{\Phi}'$ has constant order of magnitude in the unit cube, as follows from the discussion of Φ, $\tilde{\Phi}$ in the section on Geometry. Hence

$$\|\tilde{U}\|_{C^3(|y|<1)}\leqslant\frac{C}{\gamma^m}\|U\|_{C^3(\mathscr{B})}$$

$$\|\tilde{U}\|_{L^2(|y|<1)}\sim\frac{\|U\|_{L^2(\mathscr{B})}}{\sqrt{\operatorname{vol}\mathscr{B}}}.$$

Also,

$$\begin{aligned}
\|\tilde{F}_\nu\|_{L^\infty(|y|<1)}&\sim\gamma\,\|F_\nu\|_{L^\infty(\mathscr{B})}\qquad(\nu\geqslant 1)\\
\|\tilde{F}_0\|_{L^\infty(|y|<1)}&\sim\gamma^2\,\|F_0\|_{L^\infty(\mathscr{B})}\\
\|\tilde{U}\|_{L^\infty(|y|<1)}&\sim\|U\|_{L^\infty(\mathscr{B})}\\
\tilde{Y}\tilde{U}(0)&=\gamma YU(x_0)\\
\tilde{Y}\tilde{Y}'\tilde{U}(0)&=\gamma^2 YY'U(x_0).
\end{aligned}$$

Thus Lemmas B′–E′ are immediate consequences of Lemma B–E.

A variant of Lemma A is as follows.

Lemma A*. *If* $(L+B)U=F_0+\sum_{\nu\geqslant 1}\varkappa_\nu F_\nu+YY'F_*$ *in* $\{|y|<1\}$ *then*

$$|U(0)|\leqslant C\,\|F_0\|_{L^\infty}+C\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty}+C\,|\ln\eta|\,\|F_*\|_{L^\infty}$$
$$+C\,\|U\|_{-s}+C\eta^2\,\|YY'F_*\|_{L^\infty},$$

where the norms L^∞ *and* H^{-s} *refer to* $\{|y|<1\}$, *and the constants C depend on B, s.*

Sketch of Proof. The proof of Lemma A was based on two ingredients: the properties of the fundamental solution of L (proved in [FS]), and the hypoelliptic estimate

$$\|w\|_{C^3(|y|<1/4)}\leqslant C\,\|w\|_{L^2(|y|<1/2)}$$

for solutions of $Lw=0$ in $|y|<1/2$. However, the results of [FS] apply just as well to the fundamental solution of $L+B$. (Of course, the constants in the estimates will then depend on B.) Moreover, the hypoelliptic estimate holds in the sharper form

$$\|w\|_{C^3(|y|<1/4)} \leqslant C_{sB} \|w\|_{H^{-s}(|y|<1/2)}$$

for solutions of $(L+B)w=0$ in $|y|<1/2$.

Hence the proof of Lemma A may simply be copied with minor changes to prove Lemma A*. ∎

6. Fourier Transform Lemma

Fourier Transform Lemma. *Let Φ: $\prod_{k=1}^{n} I_k^+ \to \mathbb{R}^n$, where $|I_k^+| = \delta_k^+ > \delta^{1-\varepsilon}$, $|(\Phi')^{\pm 1}| < C$ on $\prod_k I_k^+$, and $|\partial^\alpha \Phi'| \leqslant C_\alpha \prod_k (\delta_k^+)^{-\alpha_k}$ (natural estimates). Let $\sigma \in C_0^\infty(\prod_k I_k^+)$ with $|\partial^\alpha \sigma| \leqslant C_\alpha \prod_k (\delta_k^+)^{-\alpha_k}$. Let $\theta(\eta)$ be a symbol of order 0, supported in $\{|\eta| \leqslant c_0(M/\delta)\}$, with $c_0 \ll 1$ depending on C, C_α, ε given above. Set $v(y)=\sigma(y)\cdot(\Gamma_\delta u)(\Phi(y))$. Then $\|\theta(D)v\| \leqslant C(p, M, s)\,\delta^p \|u\|_{-s}$.*

Proof. $\hat{v}(\eta) = \int e^{-i\eta\cdot y}\sigma(y)\, e^{i\xi\cdot\Phi(y)}\varphi((\delta/M)\xi)\,\hat{u}(\xi)\,d\xi\,dy$, with $\varphi((\delta/M)\xi)$ = symbol of Γ_δ, i.e.,

$$\hat{v}(\eta) = \int\left[\int e^{i[\xi\cdot\Phi(y)-\eta\cdot y]}\sigma(y)\,dy\right]\hat{u}(\xi)\,\varphi\left(\frac{\delta}{M}\xi\right)d\xi.$$

Fix $\eta \in \operatorname{supp}\theta$ and ξ with $|\xi| \sim \delta/M$. Note that $(\partial/\partial y_j)[\xi\cdot\Phi(y)-\eta\cdot y] = \xi\cdot\nabla_j\Phi(y)-\eta_j$. At $y_0 \in \prod_k I_k^+$, the column vectors $\nabla_j\Phi(y_0)$ together form a matrix $\Phi'(y_0)$ with $|(\Phi'(y_0))^{\pm 1}| \leqslant C$. So if $|\xi| \sim M/\delta$, then there is some j for which $|\xi\cdot\nabla_j\Phi(y_0)| \geqslant 4c_0(M/\delta)$. Our estimates for $|\partial^\alpha\Phi'|$ show that $|\xi\cdot\nabla_j\Phi(y)| \geqslant 3c_0(M/\delta)$ for all $y=(y_1, ..., y_k)$ near $y_0=(y_1^0, ..., y_n^0)$ in the sense that $|y_i - y_i^0| < c_1(\delta_i^+)$. So if $\operatorname{supp}\sigma \subset \{y \mid |y_k - y_k^0| < c_1(\delta_k^+)\} = U$ then for $|\xi| \sim M/\delta$ we can pick out j for which $|\xi\cdot\nabla_j\Phi(y)-\eta_j| < c_0(M/\delta)$ for $y \in \operatorname{supp}\sigma$, $|\eta| < c_0(M/\delta)$. By making a partition of unity, we may assume $\operatorname{supp}\sigma \subset U$. Now $F(y) \equiv ((\delta/M)\xi)\cdot\nabla_j\Phi(y) - ((\delta/M)\eta_j)$ satisfies $|\partial^\alpha F| \leqslant C_\alpha \prod_{k=1}^n (\delta_k^+)^{-\alpha_k}$ on $\prod_k I_k^+$, and moreover $|F| \geqslant c_0$ on $\operatorname{supp}\sigma$. Hence $1/F(y) = (M/\delta)\cdot(\xi\cdot\nabla_j\Phi(y)-\eta_j)$ satisfies $|\partial^\alpha(1/F(y))| \leqslant C_\alpha \prod_{k=1}^n (\delta_k^+)^{-\alpha_k}$ on $\operatorname{supp}\sigma$.

We note that $e^{i[\xi\cdot\Phi(y)-\eta\cdot y]} = (\delta/M)^N [(1/F)(\partial/\partial y_j)]^N e^{i[\xi\cdot\Phi(y)-\eta\cdot y]}$, so integration by parts yields

$$\int e^{i[\xi\cdot\Phi(y)-\eta\cdot y]}\sigma(y)\,dy = \pm\left(\frac{\delta}{M}\right)^N \int e^{i[\xi\cdot\Phi(y)-\eta\cdot y]}\left\{\left[\left(\frac{\partial}{\partial y_j}\right)\frac{1}{F}\right]^N \sigma(y)\right\}dy.$$

The expression in curly brackets is a sum of terms $\prod_l [(\partial/\partial y_j)^{s_l} F] \cdot [(\partial/\partial y_j)^t \sigma]/F^{\text{power}}$ with $\sum_l s_l + t = N$. Our estimates for $|\partial^\alpha F|$, $|\partial^\alpha \sigma|$ and the lower bound $|F| \geqslant c_0$ on $\operatorname{supp} \sigma$ therefore prove that $|[(\partial/\partial y_j)(1/F)]^N \sigma| \leqslant C_N \delta_j^{-N}$. Putting this into the above integral identity yields

$$\left| \int e^{i[\xi \cdot \Phi(y) - \eta \cdot y]} \sigma(y)\, dy \right| \leqslant C_N \left(\frac{\delta}{M}\right)^N \delta_j^{-N}$$
$$\leqslant C'_N \delta^{\varepsilon N},$$

since $\delta_j \geqslant \delta^{1-\varepsilon}$. Therefore our formula for $\hat{v}(\eta)$ implies the estimate $|\hat{v}(\eta)| \leqslant \int C'_N \delta^{\varepsilon N} |\hat{u}(\xi)|\, |\varphi((\delta/M)\xi)|\, d\xi$. Taking N large enough, we get from Cauchy–Schwartz $|\hat{v}(\eta)|^2 \leqslant C_{p,s} \delta^p \|u\|^2_{-s}$ for $|\eta| < c_0 M/\delta$. Hence

$$\|\theta(D) v\|^2 \leqslant C_{p,s} \delta^p \|u\|^2_{-s} \cdot (\text{vol supp } \theta) \leqslant C'_{p,s,M} \delta^{p-n} \|u\|^2_{-s}.$$

Since p is arbitrarily large, the lemma is proved. ∎

7. Rescaled Subelliptic Estimates

Suppose $\tilde{L} = -\sum_{jk} (\partial/\partial \tilde{x}_j)\, \tilde{a}_{jk} (\partial/\partial \tilde{x}_k)$ is self-adjoint, real, C^∞, subelliptic in $\prod_{k=1}^n \{|\tilde{x}_k| \leqslant 10\}$, with $(\tilde{a}_{jk}) \geqslant 0$. Let $\tilde{\sigma} \in C_0^\infty(\prod_k \{|\tilde{x}_k| \leqslant 1\})$, and let $\tilde{\theta}(\tilde{\eta})$ be a symbol of order zero, with

$$\tilde{\theta}(\tilde{\eta}) = \begin{cases} 1 & \text{for } |\tilde{\eta}| \leqslant \frac{1}{2} c_0 M \\ 0 & \text{for } |\tilde{\eta}| \geqslant c_0 M \end{cases}.$$

Let $\tilde{Y}$ be a vector field, subunit for $\tilde{L}$, but not necessarily smooth. Then for $\tilde{u} \in L^2(\mathbb{R}^n)$ we have the following estimate:

Lemma.

$$M^{2\varepsilon} \|\tilde{\sigma}\tilde{u}\|^2 + \|\tilde{Y}(\tilde{\sigma}\tilde{u})\|^2 \leqslant C\langle \tilde{L}\tilde{\sigma}\tilde{u}, \tilde{\sigma}\tilde{u}\rangle + M^{2\varepsilon} \|\tilde{\theta}(D_{\tilde{y}})\tilde{u}\|^2$$
$$+ C_m \int_{\mathbb{R}^n} \left(|\tilde{u}(\tilde{y})|^2\, d\tilde{y} \Big/ \left(1 + \sum_k |\tilde{y}_k|\right)^m \right).$$

Proof. Let $\tilde{\sigma}_1 \in C_0^\infty(\{|x| \leqslant 2\})$ with $\tilde{\sigma}_1 \tilde{\sigma} = \tilde{\sigma}$. The subellipticity of $\tilde{L}$ means $\|\tilde{\sigma}_1 \tilde{w}\|^2_\varepsilon \leqslant C\langle \tilde{L}\tilde{w}, \tilde{w}\rangle + C\|\tilde{w}\|^2$. Applying this to $\tilde{w} = \tilde{\sigma} u$ gives $\|\tilde{\sigma}\tilde{u}\|^2_\varepsilon \leqslant C\langle \tilde{L}\tilde{\sigma}\tilde{u}, \tilde{\sigma}\tilde{u}\rangle + C\|\tilde{\sigma}\tilde{u}\|^2$. Now let $\theta^\# \in S^0$ be supported in $\{|\tilde{\xi}| < c_0 M/2\}$, with $\theta^\# = 1$ in $\{|\tilde{\xi}| \leqslant c_0 M/4\}$. Then $M^{2\varepsilon} \|\tilde{w}\|^2 \leqslant C\|\Lambda^\varepsilon \tilde{w}\|^2 + CM^{2\varepsilon} \|\theta^\#(D)\tilde{w}\|^2$ as one sees at once from the Fourier transform. Again taking $\tilde{w}=\tilde{\sigma}\tilde{u}$, we have $M^{2\varepsilon}\|\tilde{\sigma}\tilde{u}\|^2 \leqslant C\|\tilde{\sigma}\tilde{u}\|_\varepsilon^2 + CM^{2\varepsilon}\|\theta^{\#}(D)\,\tilde{\sigma}\tilde{u}\|^2$. Hence

$$M^{2\varepsilon}\|\tilde{\sigma}\tilde{u}\|^2 \leqslant C\langle \tilde{L}\tilde{\sigma}\tilde{u}, \tilde{\sigma}\tilde{u}\rangle + C\|\tilde{\sigma}\tilde{u}\|^2 \\ + CM^{2\varepsilon}\|\theta^{\#}(D)\,\tilde{\sigma}\tilde{u}\|^2.$$

Next,

$$\|\theta^{\#}(D)\,\tilde{\sigma}\tilde{u}\|^2 \leqslant C\|\theta^{\#}(D)\,\tilde{\sigma}\tilde{\theta}(D)u\|^2 + C\|\theta^{\#}(D)\,\tilde{\sigma}(I-\tilde{\theta}(D))\tilde{u}\|^2 \\ \leqslant C\|\tilde{\theta}(D)\tilde{u}\|^2 + C\|\theta^{\#}(D)\,\tilde{\sigma}(I-\tilde{\theta}(D))\tilde{u}\|^2,$$

so

$$M^{2\varepsilon}\|\tilde{\sigma}\tilde{u}\|^2 \leqslant C\langle \tilde{L}\tilde{\sigma}\tilde{u}, \tilde{\sigma}\tilde{u}\rangle + C\|\tilde{\sigma}\tilde{u}\|^2 + CM^{2\varepsilon}\|\tilde{\theta}(D)\tilde{u}\|^2 \\ + CM^{2\varepsilon}\|\theta^{\#}(D)\,\tilde{\sigma}(I-\tilde{\theta}(D))\tilde{u}\|^2.$$

The second term on the right-hand side may be absorbed into the left-hand side. Moreover, $T=\theta^{\#}(D)\,\tilde{\sigma}(I-\tilde{\theta}(D))$ is a composite of pseudodifferential operators whose symbols have disjoint supports, with $\operatorname{supp}\tilde{\sigma}\subset\{|\tilde{x}|\leqslant 1\}$ and $\operatorname{supp}(1-\tilde{\theta})\subset\{|\xi|\geqslant(1/4)\,c_0M\}$. Hence $Tu(x)=\int K(x,y)\,u(y)\,dy$ with $|K(x,y)|\leqslant C_m(1+|x|+|y|)^{-m}$. Hence $\|\theta^{\#}(D)\,\tilde{\sigma}(I-\tilde{\theta}(D))\tilde{u}\|^2 \leqslant C_m\int_{\mathbb{R}^n}(|\tilde{u}(\tilde{y})|^2\,d\tilde{y}/(1+|\tilde{y}|)^m)$, so the lemma is proved. ∎

We apply the above lemma to the operator $\tilde{L}$ defined in Section 3 by pulling $\gamma^2 L$ back to rescaled straightened coordinates via the map $\tilde{\Phi}$. We take $\tilde{u}=\tilde{\chi}\cdot(u\circ\tilde{\Phi})$, where $\tilde{\chi}\in C_0^\infty(\prod_k\{|\tilde{x}_k|<\gamma^{-\varepsilon}\})$, $\tilde{\chi}=1$ in $\{|\tilde{x}_k|<(1/2)\gamma^{-\varepsilon}\}$ and $|(\partial/\partial\tilde{x})^\alpha\tilde{\chi}|\leqslant C_\alpha\gamma^{\varepsilon|\alpha|}$. (The cutoff χ is introduced because $\tilde{\Phi}$ is only defined on $\prod_k\{|\tilde{x}_k|<\gamma^{-\varepsilon}\}$.)

Thus, $\tilde{L}$, $\tilde{u}$ are related to L, u by $\tilde{L}\tilde{u}=\gamma^2(Lu)\circ\tilde{\Phi}$ in $\prod_k\{|\tilde{x}_k|<(1/2)\gamma^{-\varepsilon}\}$. Similarly, if Y is a (nonsmooth) subunit field for L, then γY pulls back to $\tilde{Y}$, a subunit vector field for $\tilde{L}$, with $\tilde{Y}\tilde{u}=\gamma(Yu)\circ\tilde{\Phi}$ in $\prod_k\{|\tilde{x}_k|\leqslant(1/2)\gamma^{-\varepsilon}\}$. Applying the previous lemma to $\tilde{L}$, $\tilde{u}$, $\tilde{Y}$, we obtain with

$$\sigma=\begin{cases}\tilde{\sigma}\circ\tilde{\Phi}^{-1} & \text{in } \tilde{\Phi}(\prod_k\{|\tilde{x}_k|<\gamma^{-\varepsilon}\})\\ 0 & \text{elsewhere}\end{cases},$$

the estimate

$$M^{2\varepsilon}\|(\sigma u)\circ\tilde{\Phi}\|^2+\gamma^2\|[Y(\sigma u)]\circ\tilde{\Phi}\|^2 \\ \leqslant C\gamma^2\langle[L(\sigma u)]\circ\tilde{\Phi}, (\sigma u)\circ\tilde{\Phi}\rangle + M^{2\varepsilon}\|\tilde{\theta}(D_{\tilde{y}})\{\tilde{\chi}[u\circ\tilde{\Phi}]\}\|^2 \\ + C_m\int_{\mathbb{R}^n}\frac{|\tilde{\chi}(u\circ\tilde{\Phi})(\tilde{y})|^2\,d\tilde{y}}{(1+\sum_k|\tilde{y}_k|)^m}.$$

Let us rewrite this estimate in terms of the new coordinates $\bar{y}=(\bar{y}_1,\ldots,\bar{y}_n)$, given in terms of $\tilde{y} = (\tilde{y}_1, ..., \tilde{y}_n)$ by $\bar{y}_k = \delta_k \tilde{y}_k$. Note that $u \circ \tilde{\Phi}(\tilde{y}) = u \circ \Phi(\bar{y})$, where $\tilde{\Phi}$, Φ are the straigthening maps defined in Section 3.

We introduce the cutoff functions

$$\bar{\theta}(\bar{\eta}_1, ..., \bar{\eta}_n) = \tilde{\theta}(\delta_1 \bar{\eta}_1, ..., \delta_n \bar{\eta}_n) \qquad \text{and} \qquad \bar{\chi}(\bar{y}_1, ..., \bar{y}_n) = \tilde{\chi}\left(\frac{\bar{y}_1}{\delta_1}, ..., \frac{\bar{y}_n}{\delta_n}\right).$$

The previous estimate may now be rewritten as

$$\begin{aligned} &M^{2\varepsilon} \|(\sigma u) \circ \Phi\|^2 + \gamma^2 \|[Y(\sigma u)] \circ \Phi\|^2 \\ &\quad \leqslant C\gamma^2 \langle [L(\sigma u)] \circ \Phi, (\sigma u) \circ \Phi \rangle + M^{2\varepsilon} \|\bar{\theta}(D_{\bar{y}})\{\bar{\chi}[u \circ \Phi]\}\|^2 \\ &\quad\quad + C_m \int_{\mathbb{R}^n} \frac{|\bar{\chi}(u \circ \Phi)(\bar{y})|^2 \, d\bar{y}}{(1 + \sum_k (|\bar{y}_k|/\delta_k))^m}. \end{aligned}$$

Since $\det \nabla\Phi = 1$, this means

$$\begin{aligned} &M^{2\varepsilon} \|\sigma u\|^2 + \gamma^2 \|Y(\sigma u)\|^2 \\ &\quad \leqslant C\gamma^2 \langle L(\sigma u), (\sigma u) \rangle + M^{2\varepsilon} \|\bar{\theta}(D_{\bar{y}})\{\bar{\chi}[u \circ \Phi]\}\|^2 \\ &\quad\quad + C_m \int_{\mathbb{R}^n} \frac{|\bar{\chi}(u \circ \Phi)(\bar{y})|^2 \, d\bar{y}}{(1 + \sum_k |\bar{y}_k|/\delta_k)^m}. \end{aligned}$$

We now take $\gamma = \gamma(x, \delta)$ and invoke the Corollary to Lemma 2 in Section 4 to control the last term on the right-hand side. Thus we obtain

$$\begin{aligned} &M^{2\varepsilon} \gamma(x, \delta)^{-2} \|\sigma u\|^2 + \|Y(\sigma u)\|^2 \\ &\quad \leqslant C\langle L(\sigma u), (\sigma u) \rangle + M^{2\varepsilon} \gamma(x, \delta)^2 \|\bar{\theta}(D_{\bar{y}})\{\bar{\chi}[u \circ \Phi]\}\|^2 \\ &\quad\quad + C_s \|u\|^2_{\delta, \lambda, s} \operatorname{vol} B_L(x, \gamma(x, \delta))(\gamma(x, \delta)^{-2} + M^{-\varepsilon} |\lambda|)^{-s+1}. \end{aligned}$$

Take here $\Gamma_\delta u$ in place of u. The Fourier Transform Lemma shows that

$$\|\bar{\theta}(D_{\bar{y}})\{\bar{\chi}[(\Gamma_\delta u) \circ \Phi]\}\|^2 \leqslant C_{p, s_0} \delta^p \|u\|^2_{-s_0},$$

so we obtain

$$\begin{aligned} &M^{2\varepsilon} \gamma(x, \delta)^{-2} \|\sigma \Gamma_\delta u\|^2 + \|Y(\sigma \Gamma_\delta u)\|^2 \\ &\quad \leqslant C\langle L(\sigma \Gamma_\delta u), (\sigma \Gamma_\delta u) \rangle \\ &\quad\quad + C_s \|\Gamma_\delta u\|^2_{\delta, \lambda, s} \cdot \operatorname{vol} B_L(x, \gamma(x, \delta)) \\ &\quad\quad \cdot (\gamma^{-2}(x, \delta) + M^{-\varepsilon} |\lambda|)^{-s+1} \\ &\quad\quad + C'_{p, s_0} \delta^p \|u\|^2_{-s_0}. \end{aligned}$$

Next, let $\tilde{S}$ be a first order self-adjoint pseudodifferential operator with non-negative principal symbol $\tilde{S}(x, \xi)$ supported in $\{|\xi| \sim M/\delta\}$ so that $S = \tilde{S}$ in support of the symbol of Γ_δ. Garding's inequality for $\tilde{S}$ gives

$$\langle \tilde{S}(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle \geqslant -C \|\sigma\Gamma_\delta u\|^2.$$

Adding this to the previous estimate and supposing $M^{2\varepsilon} \gg C$, we get

$$\begin{aligned}
M^{2\varepsilon}\gamma(x,\delta)^{-2} &\|\sigma\Gamma_\delta u\|^2 + \|Y(\sigma\Gamma_\delta u)\|^2 \\
&\leqslant C\langle (L+\tilde{S})(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle \\
&\quad + C_s \|\Gamma_\delta u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x, \gamma(x,\delta)) \\
&\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{-s+1} \\
&\quad + C_{p,s_0}\delta^p \|u\|^2_{-s_0}.
\end{aligned} \tag{1}$$

Now suppose $|\operatorname{Im}\lambda| \geqslant c|\lambda|$. Then self-adjointness of $L + \tilde{S}$ gives

$$|\operatorname{Im}\langle (L+\tilde{S}-\lambda)(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle| = |\operatorname{Im}\lambda| \, \|\sigma\Gamma_\delta u\|^2 \geqslant c|\lambda| \, \|\sigma\Gamma_\delta u\|^2$$

and hence

$$\begin{aligned}
\langle (L+\tilde{S})&(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle \\
&\leqslant \operatorname{Re}\langle (L+\tilde{S}-\lambda)(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle + |\lambda| \, \|\sigma\Gamma_\delta u\|^2 \\
&\leqslant C\,|\langle (L+\tilde{S}-\lambda)(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle|.
\end{aligned}$$

Thus, a fortiori

$$\begin{aligned}
|\lambda| \, \|\sigma\Gamma_\delta u\|^2 &+ \langle (L+\tilde{S})(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle \\
&\leqslant C\,|\langle (L+\tilde{S}-\lambda)(\sigma\Gamma_\delta u), \sigma\Gamma_\delta u\rangle| \\
&\quad + M^\varepsilon\gamma^{-2}(x,\delta) \|\sigma\Gamma_\delta u\|^2.
\end{aligned} \tag{2}$$

Clearly, this is true also if we assume $|\lambda| \leqslant C$ instead of $|\operatorname{Im}\lambda| \geqslant c|\lambda|$. From (1) and (2) now follows the estimate

$$\begin{aligned}
(M^{2\varepsilon}\gamma^{-2}(x,\delta) + |\lambda|) &\|\sigma\Gamma_\delta u\|^2 + \|Y(\sigma\Gamma_\delta u)\|^2 \\
&\leqslant C\,|\langle (L+\tilde{S}-\lambda)\,\sigma\Gamma_\delta u, \sigma\Gamma_\delta u\rangle| \\
&\quad + C_s \|\Gamma_\delta u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x, \gamma(x,\delta)) \\
&\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\
&\quad + C_{p,s_0}\delta^p \|u\|^2_{-s_0}.
\end{aligned}$$

Next write

$$\begin{aligned}
&|\langle (L+\tilde{S}-\lambda)(\sigma\Gamma_\delta u), (\sigma\Gamma_\delta u)\rangle| \\
&\quad\leqslant |\langle \sigma\Gamma_\delta(L+\tilde{S}-\lambda)u, (\sigma\Gamma_\delta u)\rangle| \\
&\qquad + |\langle [L+\tilde{S}, \sigma\Gamma_\delta]u, \sigma\Gamma_\delta u\rangle| \\
&\quad\leqslant |\langle \sigma\Gamma_\delta(L+S-\lambda)u, \sigma\Gamma_\delta u\rangle| \\
&\qquad + \|\sigma\Gamma_\delta(S-\tilde{S})u\|^2 + \|\sigma\Gamma_\delta u\|^2 \\
&\qquad + |\langle [L+\tilde{S}, \sigma\Gamma_\delta u]u, \sigma\Gamma_\delta u\rangle|,
\end{aligned}$$

and substitute this into the preceding estimate. Since the term $\|\sigma\Gamma_\delta u\|^2$ may be absorbed into the left-hand side, and since

$$\|\sigma\Gamma_\delta(S-\tilde{S})u\|^2 \leqslant \|\Gamma_\delta(S-\tilde{S})u\|^2 \leqslant C_{s_0,p}\,\delta^p\,\|u\|^2_{-s_0},$$

we obtain

$$\begin{aligned}
&(M^{+2\varepsilon}\gamma^{-2}(x,\delta)+|\lambda|)\,\|\sigma\Gamma_\delta u\|^2 + \|Y(\sigma\Gamma_\delta u)\|^2 \\
&\quad\leqslant C\,|\langle \sigma\Gamma_\delta(A-\lambda)u, \sigma\Gamma_\delta u\rangle| \\
&\qquad + C\,|\langle [L+\tilde{S}, \sigma\Gamma_\delta]u, \sigma\Gamma_\delta u\rangle| \\
&\qquad + C_s\,\|\Gamma_\delta u\|^2_{\delta,\lambda,s}\,\mathrm{vol}\,B_L(x,\gamma(x,\delta)) \\
&\qquad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{1-s} \\
&\qquad + C_{p,s_0}\,\delta^p\,\|u\|^2_{-s_0}.
\end{aligned}$$

(Recall $A = L+S$.)

We rewrite this slightly. We have

$$\begin{aligned}
c\,\|\sigma Y\Gamma_\delta u\|^2 &\leqslant \|Y(\sigma\Gamma_\delta u)\|^2 + \|(Y\sigma)\,\Gamma_\delta u\|^2 \\
&\leqslant \|Y(\sigma\Gamma_\delta u)\|^2 + C\gamma(x,\delta)^{-2}\,\|\Gamma_\delta u\|^2_{L^2(B(x,\gamma(x,\delta)))} \\
&\leqslant \|Y(\sigma\Gamma_\delta u)\|^2 + C_s\,\|\Gamma_\delta u\|^2_{\delta,\lambda,s}\,\mathrm{vol}\,B_L(x,\gamma(x,\delta)) \\
&\qquad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{1-s}
\end{aligned}$$

and also

$$\begin{aligned}
|\langle [L+\tilde{S}, \sigma\Gamma_\delta]u, \sigma\Gamma_\delta u\rangle| &\leqslant \|\sigma\Gamma_\delta u\|^2\,(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)M^{+\varepsilon/10} \\
&\quad + \|[L+\tilde{S}, \sigma\Gamma_\delta]u\|^2_{L^2(\mathrm{supp}\,\sigma)} \\
&\quad \times(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{-1}\cdot M^{-\varepsilon/10} \\
&\leqslant C_s\,\|\Gamma_\delta u\|^2_{\delta,\lambda,s}\,\mathrm{vol}\,B_L(x,\gamma(x,\delta)) \\
&\quad \times(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{1-s}M^{+\varepsilon/10} \\
&\quad + (\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{-1} \\
&\quad \times\|[L+\tilde{S}, \sigma\Gamma_\delta]u\|^2_{L^2(\mathrm{supp}\,\sigma)}\,M^{-\varepsilon/10}.
\end{aligned}$$

Putting these into our previous estimate gives

$$\begin{aligned}
(M^{2\varepsilon}\gamma^{-2}(x,\delta)+|\lambda|)\,\|\sigma\Gamma_\delta u\|^2+\|\sigma Y\Gamma_\delta u\|^2
&\leqslant C\,|\langle\sigma\Gamma_\delta(A-\lambda)u,\sigma\Gamma_\delta u\rangle| \\
&\quad + C(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{-1} \\
&\quad \times \|[L+\tilde{S},\sigma\Gamma_\delta]u\|^2_{L^2(\operatorname{supp}\sigma)}\,M^{-\varepsilon/10} \\
&\quad + C_s\,\|\Gamma_\delta u\|^2_{\delta,\lambda,s}\operatorname{vol}B_L(x,\gamma(x,\delta)) \\
&\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{1-s}\,M^{\varepsilon/10} \\
&\quad + C_{p,s_0}\,\delta^p\,\|u\|^2_{-s_0}.
\end{aligned} \tag{3}$$

Next we estimate the commutator term in (3).

8. Commutator Terms

We have $[L+\tilde{S},\ \sigma\Gamma_\delta]u=[L,\ \sigma]\,\Gamma_\delta u+[\tilde{S},\sigma]\,\Gamma_\delta u+\sigma[L,\Gamma_\delta]u+\sigma[\tilde{S},\Gamma_\delta]u$. Now $[L,\sigma]=C\gamma(x,\delta)^{-1}\varkappa+(L\sigma)$ with $\varkappa$ subunit for L (but not smooth) and $|L\sigma|\leqslant C\gamma(x,\delta)^{-2}$. (This is easily seen by studying $[\gamma^2 L,\sigma]$ in rescaled straightened coordinates.) Hence

$$\begin{aligned}
\|[L,\sigma]\,\Gamma_\delta u\|^2_{L^2(\operatorname{supp}\sigma)} &\leqslant C\gamma(x,\delta)^{-2}\,\|\varkappa\Gamma_\delta u\|^2_{L^2(\operatorname{supp}\sigma)} \\
&\quad + C\gamma(x,\delta)^{-4}\,\|\Gamma_\delta u\|^2_{L^2(\operatorname{supp}\sigma)} \\
&\leqslant C\,\|\varkappa\Gamma_\delta u\|^2_{\delta,\lambda,s-1}\operatorname{vol}B_L(x,\gamma(x,\delta)) \\
&\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{1-s}\,\gamma(x,\delta)^{-2} \\
&\quad + C\,\|\Gamma_\delta u\|^2_{\delta,\lambda,s}\operatorname{vol}B_L(x,\gamma(x,\delta)) \\
&\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{-s}\,\gamma(x,\delta)^{-4} \\
&\leqslant C\,\|\Gamma_\delta u\|^2_{\delta,\lambda,s}\operatorname{vol}B_L(x,\gamma(x,\delta)) \\
&\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{2-s} \\
&\quad + C\,\|\varkappa\Gamma_\delta u\|^2_{\delta,\lambda,s-1}\operatorname{vol}B_L(x,\gamma(x,\delta)) \\
&\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{2-s}.
\end{aligned} \tag{C1}$$

Next recall $\tilde{S}=\lambda(x)Q$ where Q is a first order symbol, and $|\lambda(y)|\leqslant C\,\delta\gamma^{-2}(x,\delta)$ in $B_L(x,\gamma(x,\delta))$. Hence $\tilde{S}=\lambda(x)\tilde{Q}$ with $\tilde{Q}(x,\xi)$ supported in $|\xi|\sim M/\delta$, i.e., $\tilde{S}=(M/\delta)\,\lambda(x)\cdot Q^{\#}$ with $Q^{\#}$ a symbol of order zero, supported in $\{|\xi|\sim M/\delta\}$. Thus $[\tilde{S},\sigma]=(M/\delta)\,\lambda(y)\cdot[Q^{\#},\sigma]$. Since $|\lambda(y)|/\delta\leqslant C\gamma^{-2}(x,\delta)$ for $y\in\operatorname{supp}\sigma$, we have

$$\begin{aligned}\|[\tilde{S}, \sigma]\, \Gamma_\delta u\|^2_{L^2(\operatorname{supp} \sigma)} &\leqslant C\gamma^{-4}(x,\delta)\, \|M[Q^\#, \sigma]\, \Gamma_\delta u\|^2_{L^2(\operatorname{supp}\sigma)} \\ &\leqslant C\gamma^{-4}(x,\delta)\cdot \|M[Q^\#,\sigma]\,\Gamma_\delta u\|^2_{\delta,\lambda,s} \\ &\quad \times \operatorname{vol} B_L(x,\gamma(x,\delta))\cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}\,|\lambda|)^{-s} \\ &\leqslant C'\gamma^{-4}(x,\delta)\cdot \|\Gamma_\delta u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}\,|\lambda|)^{-s}\end{aligned}$$

(by virtue of (c) in Lemma 4 in Section 4)

$$\leqslant C'\; \|\Gamma_\delta u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x,\gamma(x,\delta))\cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}\,|\lambda|)^{2-s}. \qquad \text{(C2)}$$

Next look at $\sigma[L, \Gamma_\delta]u$. Now $L=\sum_j X_j^*\phi_j X_j$ with $\phi_j \geqslant 0$. Both $\phi_j X_j$ and $(\partial\phi_j/\partial x_k)X_j$ are subunit, the latter since $|\nabla\phi_j|^2 \leqslant C\phi_j$. Hence $[L, \Gamma_\delta] = \sum_{j\geqslant 1} Q_j\tilde{X}_j + Q_0 + Q_{\text{error}}$ where Q_j, Q_0 are pseudodifferential operators of order 0 supported in $\{|\xi| \sim M/\delta\}$ and $\|Q_{\text{error}}\, u\| \leqslant C_{p,s_0}\,\delta^p\, \|u\|_{-s_0}$ and $\tilde{X}_j$ subunit (see the beginning of Section 9). With suitable $\tilde{\Gamma}_\delta$ analogous to Γ_δ but having symbol equal to one on the support of symbols of Q_j and Q_0, we have

$$[L, \Gamma_\delta]u = \sum_{j\geqslant 1} Q_j\tilde{X}_j\tilde{\Gamma}_\delta u + Q_0\tilde{\Gamma}_\delta u + Q_{\text{error}}\, u$$

(for a different Q_{error}, satisfying the same estimate). Hence

$$\begin{aligned}\|\sigma[L,\Gamma_\delta]u\|^2 &\leqslant C\sum_j \|\sigma Q_j\tilde{X}_j\tilde{\Gamma}_\delta u\|^2 + C\,\|Q_0\tilde{\Gamma}_\delta u\|^2 + C\,\delta^p\,\|u\|^2_{-s_0} \\ &\leqslant C\sum_j \|Q_j\tilde{X}_j\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-2} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}\,|\lambda|)^{2-s} \\ &\quad + C\,\|Q_0\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-1} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}\,|\lambda|)^{2-s} \\ &\quad + C_{p,s_0}\,\delta^p\,\|u\|^2_{-s_0}.\end{aligned}$$

(We have been wasteful with the Q_0-term.)

Invoking (c) in Lemma 4 in Section 4, we get

$$\begin{aligned}\|\sigma[L,\Gamma_\delta]u\|^2 &\leqslant C\sum_j \|\tilde{X}_j\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-2} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad \cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}\,|\lambda|)^{2-s} \\ &\quad + C\,\|\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-1} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad \cdot(\gamma^{-2}(x,\delta)M^{-\varepsilon}\,|\lambda|)^{2-s} \\ &\quad + C_{p,s_0}\,\delta^p\,\|u\|^2_{-s_0}. \qquad \text{(C3)}\end{aligned}$$

Similarly, $[\tilde{S}, \Gamma_\delta] = Q\tilde{\Gamma}_\delta + Q_{\text{error}}$ with Q of order zero with symbol supported in $\{|\xi| \sim M/\delta\}$, $\|Q_{\text{error}} u\| \leqslant C_{p,s_0} \delta^p \|u\|_{-s_0}$. So

$$\begin{aligned}\|\sigma[\tilde{S}, \Gamma_\delta]u\|^2 &\leqslant \|\sigma Q\tilde{\Gamma}_\delta u\|^2 + C_{p,s_0} \delta^p \|u\|^2_{-s_0} \\ &\leqslant \|Q\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-1} \operatorname{vol} B_L(x, \gamma(x,\delta)) \\ &\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{2-s} \\ &\quad + C_{p,s_0} \delta^p \|u\|^2_{-s_0} \qquad \text{(C4)}\end{aligned}$$

(again we are being wasteful)

$$\begin{aligned}&\leqslant C \|\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-1} \operatorname{vol} B_L(x,\gamma(x,\delta)) \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{2-s} \\ &\quad + C_{p,s_0}\delta^p \|u\|^2_{-s_0}\end{aligned}$$

(again by (a) of Lemma 4 in Section 4).

Putting (C1) $\cdots$ (C4) into the formula for $[L + \tilde{S}, \sigma\Gamma_\delta]$ we get

$$\begin{aligned}\|[L+\tilde{S}, \sigma\Gamma_\delta]u\|^2 \leqslant{} & C \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ & \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{2-s} \{[\Gamma_\delta u\|^2_{\delta,\lambda,s} \\ & + \|\mathscr{X}\Gamma_\delta u\|^2_{\delta,\lambda,s-1} + \|\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-1} + \|\mathscr{W}\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-2} \\ & + C_{p,s_0}\delta^p \|u\|^2_{-s_0}\},\end{aligned}$$

where $\mathscr{W} = \tilde{X}_{j_0}$ with j_0 picked to maximize $\|\tilde{X}_{j_0}\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-2}$. Putting this estimate into (3) from Section 7,

$$\begin{aligned}&(M^{2\varepsilon}\gamma^{-2}(x,\delta) + |\lambda|)\,\|\sigma\Gamma_\delta u\|^2 + \|\sigma Y\Gamma_\delta u\|^2 \\ &\quad \leqslant C\,|\langle \sigma\Gamma_\delta(A-\lambda)u, \sigma\Gamma_\delta u\rangle| \\ &\quad\quad + C_s M^{\varepsilon/10} \|\Gamma_\delta u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\ &\quad\quad + C_s M^{-\varepsilon/10} \|\mathscr{X}\Gamma_\delta u\|^2_{\delta,\lambda,s-1} \operatorname{vol} B_L(x,\gamma(x,\delta)) \\ &\quad\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\ &\quad\quad + C_s(\|\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-1} + \|\mathscr{W}\tilde{\Gamma}_\delta u\|^2_{\delta,\lambda,s-2}) \\ &\quad\quad \cdot \operatorname{vol} B_L(x,\gamma(x,\delta)) \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\ &\quad\quad + C_{s_0 p}\delta^p \|u\|^2_{-s_0}.\end{aligned}$$

Recall Y is subunit, while $\mathscr{X}$, $\mathscr{W}$ are appropriate subunit vector fields.

Now assume $(A-\lambda)u = f_0 + \sum_{\nu \geqslant 1} Y_\nu f_\nu$ with Y_ν *smooth*, subunit. We have

$$|\langle\sigma\Gamma_\delta(A-\lambda)u, \sigma\Gamma_\delta u\rangle| \leqslant |\langle\sigma\Gamma_\delta f_0, \sigma\Gamma_\delta u\rangle| + \sum_\nu |\langle\sigma\Gamma_\delta Y_\nu f_\nu, \sigma\Gamma_\delta u\rangle| \tag{C6}$$

and

$$\begin{aligned}\langle\sigma\Gamma_\delta Y_\nu f_\nu, \sigma\Gamma_\delta u\rangle &= \langle\sigma[\Gamma_\delta, Y_\nu] f_\nu, \sigma\Gamma_\delta u\rangle + \langle[\sigma, Y_\nu]\, \Gamma_\delta f_\nu, \sigma\Gamma_\delta u\rangle \\ &\quad + \langle Y_\nu \sigma\Gamma_\delta f_\nu, \sigma\Gamma_\delta u\rangle \\ &= \langle\sigma[\Gamma_\delta, Y_\nu] f_\nu, \sigma\Gamma_\delta u\rangle + \langle[\sigma, Y_\nu]\, \Gamma_\delta f_\nu, \sigma\Gamma_\delta u\rangle \\ &\quad + \langle\sigma\Gamma_\delta f_\nu, [Y_\nu^*, \sigma]\, \Gamma_\delta u\rangle + \langle\sigma\Gamma_\delta f_\nu, \sigma Y_\nu^* \Gamma_\delta u\rangle.\end{aligned} \tag{C7}$$

Let us estimate the size of the terms in (C7).

$$\begin{aligned}|\langle\sigma[\Gamma_\delta, Y_\nu] f_\nu, \sigma\Gamma_\delta u\rangle| &\leqslant (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)\, \|\sigma\Gamma_\delta u\|^2 \\ &\quad + (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{-1}\, \|\sigma[\Gamma_\delta, Y_\nu] f_\nu\|^2 \\ &\leqslant C_s\, \|\Gamma_\delta u\|^2_{\delta,\lambda,s} \operatorname{vol} B_L(x, \gamma(x,\delta)) \\ &\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\ &\quad + C_s\, \|[\Gamma_\delta, Y_\nu] f_\nu\|^2_{\delta,\lambda,s-1} \operatorname{vol} B_L(x, \gamma(x,\delta)) \\ &\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{-s}.\end{aligned}$$

Since $[\Gamma_\delta, Y_\nu] = Q^\# \tilde{\Gamma}_\delta + Q_{\mathrm{error}}$ with $Q^\#$ symbol of order zero supported in $\{|\xi| \sim M/\delta\}$ and $\|Q_{\mathrm{error}} f\| \leqslant C_{p,s_0}\, \delta^p\, \|f\|_{-s_0}$, we invoke (a) of Lemma 4 in Section 4 to conclude that

$$\begin{aligned}|\langle\sigma[\Gamma_\delta, Y_\nu] f_\nu, \sigma\Gamma_\delta u\rangle| &\leqslant C \operatorname{vol} B_L(x, \gamma(x,\delta)) \\ &\quad \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\ &\quad \times \{\|\Gamma_\delta u\|^2_{\delta,\lambda,s} + \|\tilde{\Gamma}_\delta f_\nu\|^2_{\delta,\lambda,s-1} + C_{p,s_0}\, \delta^p\, \|f_\nu\|^2_{-s_0}\}.\end{aligned} \tag{C8}$$

Now $[\sigma, Y_\nu]$ and $[\sigma, Y_\nu^*]$ are multiplication by a function $0(\gamma^{-1}(x,\delta))$ and supported in $\operatorname{supp}\sigma$. Hence

$$\begin{aligned}&|\langle[\sigma, Y_\nu]\, \Gamma_\delta f_\nu, \sigma\Gamma_\delta u\rangle| + |\langle\sigma\Gamma_\delta f_\nu, [Y_\nu^*, \sigma]\, \Gamma_\delta u\rangle| \\ &\quad \leqslant C\gamma^{-1}(x,\delta)\, \|\Gamma_\delta f_\nu\|_{L^2(\operatorname{supp}\sigma)}\, \|\Gamma_\delta u\|_{L^2(\operatorname{supp}\sigma)} \\ &\quad \leqslant C(\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)\, \|\Gamma_\delta u\|^2_{L^2(\operatorname{supp}\sigma)} \\ &\qquad + C\, \|\Gamma_\delta f_\nu\|^2_{L^2(\operatorname{supp}\sigma)} \\ &\quad \leqslant C \operatorname{vol} B_L(x, \gamma(x,\delta)) \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}|\lambda|)^{1-s} \\ &\qquad \times \{\|\Gamma_\delta u\|^2_{\delta,\lambda,s} + \|\Gamma_\delta f_\nu\|^2_{\delta,\lambda,s-1}\}.\end{aligned} \tag{C9}$$

Since $Y_\nu^* = -Y_\nu + b$, $b = 0(1)$, we have

$$\begin{aligned}
&|\langle \sigma\Gamma_\delta f_\nu, \sigma Y_\nu^* \Gamma_\delta u\rangle| \\
&\quad \leqslant M^{\varepsilon/10} C\, \|\sigma\Gamma_\delta f_\nu\|^2 + C\, \|\sigma Y_\nu^* \Gamma_\delta u\|^2\, M^{-\varepsilon/10} \\
&\quad \leqslant C\, \|\sigma\Gamma_\delta f_\nu\|^2\, M^{\varepsilon/10} + C\, \|\sigma Y_\nu \Gamma_\delta u\|^2\, M^{-\varepsilon/10} + C\, \|\sigma\Gamma_\delta u\|^2\, M^{-\varepsilon/10} \\
&\quad \leqslant C \operatorname{vol} B_L(x, \gamma(x,\delta))(\gamma^{-2}(x,\delta) + M^{-\varepsilon}\,|\lambda|)^{1-s} \\
&\qquad \times \{M^{\varepsilon/10}\, \|\Gamma_\delta f_\nu\|^2_{\delta,\lambda,s-1} + M^{-\varepsilon/10}\, \|Y_\nu \Gamma_\delta u\|^2_{\delta,\lambda,s-1} + \|\Gamma_\delta u\|^2_{\delta,\lambda,s}\}. \qquad \text{(C10)}
\end{aligned}$$

Putting (C8), (C9), (C10), into (C7) yields

$$\begin{aligned}
|\langle \sigma\Gamma_\delta Y_\nu f_\nu, \sigma\Gamma_\delta u\rangle| &\leqslant C \operatorname{vol} B_L(x, \gamma(x,\delta)) \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}\,|\lambda|)^{1-s} \\
&\quad \cdot \{\|\Gamma_\delta u\|^2_{\delta,\lambda,s} + \|Y_\nu \Gamma_\delta u\|^2_{\delta,\lambda,s-1}\, M^{-\varepsilon/10} \\
&\quad + \|\tilde{\Gamma}_\delta f_\nu\|^2_{\delta,\lambda,s-1}\, M^{\varepsilon/10} + C_{p,s_0}\, \delta^p\, \|f_\nu\|^2_{-s_0}\}. \qquad \text{(C11)}
\end{aligned}$$

(Here we used Lemma 4 on Norms to show that

$$\|\Gamma_\delta f_\nu\|^2_{\delta,\lambda,s-1} \leqslant C(\|\tilde{\Gamma}_\delta f_\nu\|^2_{\delta,\lambda,s-1} + C\,\delta^p\,\|f_\nu\|_{-s_0}).)$$

Also, we note that

$$\begin{aligned}
|\langle \sigma\Gamma_\delta f_0, \sigma\Gamma_\delta u\rangle| &\leqslant C(\gamma^{-2}(x,\delta) + M^{-\varepsilon}\,|\lambda|)^{-1}\, \|\sigma\Gamma_\delta f_\nu\|^2 \\
&\quad + C(\gamma^{-2}(x,\delta) + M^{-\varepsilon}\,|\lambda|)\, \|\sigma\Gamma_\delta u\|^2 \\
&\leqslant C \operatorname{vol} B_L(x, \gamma(x,\delta)) \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}\,|\lambda|)^{1-s} \\
&\quad \cdot \{\|\Gamma_\delta f_0\|^2_{\delta,\lambda,s-2} + \|\Gamma_\delta u\|^2_{\delta,\lambda,s}\}.
\end{aligned}$$

Putting this and (C11) into (C6), we obtain

$$\begin{aligned}
&|\langle \sigma\Gamma_\delta(A-\lambda)u, \sigma\Gamma_\delta u\rangle| \\
&\quad \leqslant C \operatorname{vol} B_L(x, \gamma(x,\delta)) \cdot (\gamma^{-2}(x,\delta) + M^{-\varepsilon}\,|\lambda|)^{1-s} \\
&\qquad \cdot \Big\{\|\Gamma_\delta u\|^2_{\delta,\lambda,s} + \|\Gamma_\delta f_0\|^2_{\delta,\lambda,s-2} + \sum_\nu \|Y_\nu \Gamma_\delta u\|^2_{\delta,\lambda,s-1}\, M^{-\varepsilon/10} \\
&\qquad + \sum_\nu \|\tilde{\Gamma}_\delta f_\nu\|^2_{\delta,\lambda,s-1} M^{\varepsilon/10} + \sum_\nu C_{p,s_0}\, \delta^p\, \|f_\nu\|^2_{-s_0}\Big\}.
\end{aligned}$$

We now substitute this into (C5). After possibly changing $\varkappa$ to one of the Y_ν, we obtain

$$
\begin{aligned}
&(M^{2\varepsilon}\gamma^{-2}(x,\delta)+|\lambda|)\,\|\sigma\Gamma_\delta u\|^2+\|\sigma Y\Gamma_\delta u\|^2\\
&\quad\leqslant C\operatorname{vol}B_L(x,\gamma(x,\delta))\cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{1-s}\\
&\qquad\cdot\Big\{\|\tilde\Gamma_\delta f_0\|^2_{\delta,\lambda,s-2}+\sum_{\nu\geqslant 1}\|\tilde\Gamma_\delta f_\nu\|^2_{\delta,\lambda,s-1}M^{\varepsilon/10}\\
&\qquad+M^{\varepsilon/10}\|\Gamma_\delta u\|^2_{\delta,\lambda,s}+C_sM^{-\varepsilon/10}\|\varkappa\Gamma_\delta u\|^2_{\delta,\lambda,s-1}\\
&\qquad+C_s\|\tilde\Gamma_\delta u\|^2_{\delta,\lambda,s-1}+C_s\|\mathscr{W}\tilde\Gamma_\delta u\|^2_{\delta,\lambda,s-2}\\
&\qquad+C_{p_0,s_0}\delta^p\Big(\|u\|^2_{-s_0}+\sum_\nu\|f_\nu\|^2_{-s_0}\Big)\Big\}.
\end{aligned}
$$

That is,

$$M^\varepsilon\|\sigma\Gamma_\delta u\|^2/[\operatorname{vol}B_L(x,\gamma(x,\delta))\cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{-s}]$$

and

$$\|\sigma Y\Gamma_\delta u\|^2/[\operatorname{vol}B_L(x,\gamma(x,\delta))\cdot(\gamma^{-2}(x,\delta)+M^{-\varepsilon}|\lambda|)^{-(s-1)}]$$

are bounded by the expressions in curly brackets in the preceding estimate.

This holds for all x. Taking the supremum over all x, we obtain

$$
\begin{aligned}
&[M^\varepsilon\|\Gamma_\delta u\|^2_{\delta,\lambda,s}+\|Y\Gamma_\delta u\|^2_{\delta,\lambda,s-1}]\\
&\quad\leqslant C(M)\cdot\Big[\|\tilde\Gamma_\delta f_0\|^2_{\delta,\lambda,s-2}+\sum_{\nu\geqslant 1}\|\tilde\Gamma_\delta f_\nu\|^2_{\delta,\lambda,s-1}\\
&\qquad+C_{p,s_0}\delta^p\Big(\|u\|^2_{-s_0}+\sum_{\nu\geqslant 1}\|f_\nu\|^2_{-s_0}\Big)\Big]\\
&\qquad+C[M^{\varepsilon/10}\|\Gamma_\delta u\|^2_{\delta,\lambda,s}+M^{-\varepsilon/10}\|\varkappa\Gamma_\delta u\|^2_{\delta,\lambda,s-1}]\\
&\qquad+C(M)[\|\tilde\Gamma_\delta u\|^2_{\delta,\lambda,s-1}+\|\mathscr{W}\tilde\Gamma_\delta u\|^2_{\delta,\lambda,s-2}].
\end{aligned}
$$

Defining

$$\Omega(w,\delta,\lambda,s)=M^{\varepsilon/10}\|w\|^2_{\delta,\lambda,s}+\sup_{Y\text{ subunit}}M^{-\varepsilon/10}\|Yw\|^2_{\delta,\lambda,s-1},$$

we may rewrite the above estimate as

$$
\begin{aligned}
M^{\varepsilon/10}\Omega(\Gamma_\delta u,\delta,\lambda,s)&\leqslant C(M)\cdot\Big[\|\tilde\Gamma_\delta f_0\|^2_{\delta,\lambda,s-2}+\sum_{\nu\geqslant 1}\|\tilde\Gamma_\delta f_\nu\|^2_{\delta,\lambda,s-1}\\
&\qquad+C_{p,s_0}\delta^p\Big(\sum_\nu\|f_\nu\|^2_{-s_0}+\|u\|^2_{-s_0}\Big)\Big]\\
&\qquad+C\Omega(\Gamma_\delta u,\delta,\lambda,s)+C(M)\cdot\Omega(\tilde\Gamma_\delta u,\delta,\lambda,s-1).
\end{aligned}
$$

We now pick $M \geqslant 1$ once and for all so that the term $C\Omega(\Gamma_\delta u, \delta, \lambda, s)$ on the right may be absorbed into the left-hand side.

With M fixed, we get

$$\Omega(\Gamma_\delta u, \delta, \lambda, s) \leqslant C\Omega(\tilde{\Gamma}_\delta u, \delta, \lambda, s-1) + C\Bigg[\|\tilde{\Gamma}_\delta f_0\|^2_{\delta,\lambda,s-2} + \sum_{v \geqslant 1} \|\tilde{\Gamma}_\delta f_v\|^2_{\delta,\lambda,s-1} + C_{p,s_0}\,\delta^p \left(\|u\|^2_{-s_0} + \sum_v \|f_v\|^2_{-s_0} \right) \Bigg].$$

The first term on the right-hand side is analogous to the left-hand side, but with s replaced by $s-1$. Hence we may feed this inequality into itself in an obvious induction, obtaining

$$\Omega(\Gamma_\delta u, \delta, \lambda, s) \leqslant \Omega(\tilde{\Gamma}_\delta u, \delta, \lambda, s-k) + C_k \cdot \Bigg[\|\tilde{\Gamma}_\delta f_0\|^2_{\delta,\lambda,s-2} + \sum_{v \geqslant 1} \|\tilde{\Gamma}_\delta f_v\|^2_{\delta,\lambda,s-1} + C_{p,s_0}\,\delta^p \left(\|u\|^2_{-s_0} + \sum_v \|f_v\|^2_{-s_0} \right) \Bigg].$$

If k is taken large enough, then $\Omega(\tilde{\Gamma}_\delta u, \delta, \lambda, s-k) \leqslant C_{p,s_0} \|u\|^2_{-s_0}$. Thus we may drop the first term on the right-hand side above. Recalling the definition of Ω, we obtain at last our basic localized L^2-estimate:

Suppose $(A-\lambda)u = f_0 + \sum_v Y_v f_v$ with Y_v *smooth, subunit.*

Assume $|\mathrm{Im}\,\lambda| \geqslant c\,|\lambda|$ or $|\lambda| \leqslant C$. Let Y be *any* subunit vector field. Then

$$\begin{aligned}\|\Gamma_\delta u\|_{\delta,\lambda,s} &+ \|Y\Gamma_\delta u\|_{\delta,\lambda,s-1} \\ &\leqslant C\,\|\tilde{\Gamma}_\delta f_0\|_{\delta,\lambda,s-2} + C \sum_{v \geqslant 1} \|\tilde{\Gamma}_\delta f_v\|_{\delta,\lambda,s-1} \\ &\quad + C\,\delta^p \|u\|_{-s_0} + C\,\delta^p \sum_{v \geqslant 0} \|f_v\|_{-s_0}.\end{aligned}$$

9. L^∞ Estimates

Our goal here is to estimate $(A-\lambda)^{-1}$ in the norms $|||\cdot|||_{\delta,\lambda,s}$. Again, suppose $|\lambda| \leqslant C$ or else $|\mathrm{Im}\,\lambda| \geqslant c\,|\lambda|$. We start with the equation

$$(A-\lambda)u = f_0 + \sum_{v \geqslant 1} \varkappa_v f_v \qquad \text{with} \quad \varkappa_v \text{ smooth, subunit vector fields.} \quad (0)$$

We apply Γ_δ to both sides and recall that $A = L + S$. Thus

$$(L-\lambda)u=f_0-Su+\sum_{\nu\geqslant 1}\varkappa_\nu f_\nu$$

$$\Gamma_\delta(L-\lambda)u=\left\{\Gamma_\delta f_0-S\Gamma_\delta u+[S,\Gamma_\delta]u+\sum_{\nu\geqslant 1}[\varkappa_\nu,\Gamma_\delta]f_\nu\right\}$$
$$+\sum_{\nu\geqslant 1}\varkappa_\nu(\Gamma_\delta f_\nu)$$

$$(L-\lambda)(\Gamma_\delta u)=\left\{\Gamma_\delta f_0+[L,\Gamma_\delta]u-S\Gamma_\delta u+[S,\Gamma_\delta]u+\sum_{\nu\geqslant 1}[\varkappa_\nu,\Gamma_\delta]f_\nu\right]\Bigg\}$$
$$+\sum_{\nu\geqslant 1}\varkappa_\nu(\Gamma_\delta f_\nu). \tag{1}$$

Let us examine the commutators in (1). Since S, $\varkappa_\nu$ are first-order, both $[S,\Gamma_\delta]$ and $[\varkappa_\nu,\Gamma_\delta]$ have the form $Q_0\tilde{\Gamma}_\delta+Q_{\text{error}}$ with Q_0 a pseudodifferential operator of order zero with symbol supported in $\{|\xi|\sim M/\delta\}$ and Q_{error} satisfying the estimate $\|Q_{\text{error}}\,g\|_{C^3}\leqslant C\delta^p\,\|g\|_{-s_0}$.

The commutator $[L,\Gamma_\delta]$ has the form $\sum_j Y_jQ_j+Q_0+Q_{\text{error}}$, with Q_0, Q_{error} as above, with Y_j smooth subunit vector fields and with Q_j a pseudodifferential operator of order zero with symbols supported in $\{|\xi|\sim M/\delta\}$. To see this, recall that $L=\sum_j X_j^*\phi_jX_j$ for real vector fields X_j and $\phi_j\geqslant 0$. Thus, $[L,\Gamma_\delta]$ is a sum of terms $[X_j^*,\Gamma_\delta](\phi_jX_j)$, $(X_j^*\circ\phi_j)$ $[X_j,\Gamma_\delta]$, $X_j^*[\phi_j,\Gamma_\delta]X_j$. The first two terms obviously have the right form since ϕ_jX_j is subunit. Modulo errors of order -2, $[\phi_j,\Gamma_\delta]$ has symbol $\pm i\{\phi_j(x),\Gamma_\delta(x,\xi)\}$, which is a sum of terms $(\partial\phi_j/\partial x_k)(\partial\Gamma_\delta/\partial\xi_k)$. Hence, modulo errors Q_0+Q_{error}, $X_j^*[\phi,\Gamma_\delta]X_j$ is a sum of terms

$$\left(\frac{\partial\phi_j}{\partial x_k}\right)X_j\left\{\left[\left(\frac{\partial\Gamma_\delta}{\partial\xi_k}\right)(x,D)\right]X_j^*\right\}.$$

These terms have the right form, since $c(\partial\phi_j/\partial x_k)X_j$ is subunit, as follows from the standard inequality $|\partial\phi/\partial x_k|^2\leqslant C\phi$ for non-negative smooth functions. Thus, $[L,\Gamma_\delta]$ has the form $\sum_j Y_jQ_j+Q_0+Q_{\text{error}}$.

Substituting these commutator identities into (1), we obtain

$$(L-\lambda)(\Gamma_\delta u)=\left\{\Gamma_\delta f_0+Q_0'\tilde{\Gamma}_\delta u+Q'_{\text{error}}u+\sum_{\nu\geqslant 1}Q_0^\nu\tilde{\Gamma}_\delta f_\nu\right.$$
$$\left.+\sum_{\nu\geqslant 1}Q^\nu_{\text{error}}f_\nu-S\Gamma_\delta u\right\}$$
$$+\sum_{\nu\geqslant 1}\varkappa_\nu(\Gamma_\delta f_\nu)+\sum_j Y_j(Q_j\tilde{\Gamma}_\delta u)$$
$$\equiv F_0+\sum_{\nu\geqslant 1}\mathscr{W}_\nu F_\nu. \tag{2}$$

Here $\mathscr{W}_\nu$ denotes either $\not{z}_\nu$ or Y_j; $F_\nu(\nu \geqslant 1)$ denotes either $\Gamma_\delta f_\nu$ or $Q_j \tilde{\Gamma}_\delta u$; F_0 denotes the messy expression in curly brackets; Q_0' and Q_0^ν denote pseudodifferential operators with order zero symbols supported in $\{|\xi| \sim M/\delta\}$; and $\|Q'_{\text{error}} g\|_{L^\infty}$, $\|Q^\nu_{\text{error}} g\|_{L^\infty} \leqslant C_{p,s_0} \delta^p \|g\|_{-s_0}$. All the $\mathscr{W}_\nu$ are smooth, subunit vector fields.

Now for arbitrary $x_0 \in \mathbb{R}^n$ we shall estimate the L^∞ norms of F_0, F_ν on $\mathscr{B} = B_L(x_0, \gamma(x_0, \delta))$ in terms of the norms $|||\cdot|||_{\text{etc.}}$ of u, f_0, f_ν. From the definition of the norms and from (b) of Lemma 4 in Section 4 (boundedness of pseudodifferential operators in terms of $|||\cdot|||$) we have

$$\|\Gamma_\delta f_0\|_{L^\infty(\mathscr{B})} \leqslant (\gamma(x_0, \delta)^{-2} + |\lambda|)^{1-s/2} \, |||\Gamma_\delta f_0|||_{\delta,\lambda,s-2} \tag{3}$$

$$\begin{aligned}\|Q_0' \tilde{\Gamma}_\delta u\|_{L^\infty(\mathscr{B})} &\leqslant (\gamma^{-2}(x_0, \delta)) + |\lambda|)^{1-s/2} \, |||Q_0' \tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-2} \\ &\leqslant C(\gamma^{-2}(x_0, \delta) + |\lambda|)^{1-s/2} \, |||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-2}\end{aligned} \tag{4}$$

$$\|Q'_{\text{error}} u\|_{L^\infty(\mathscr{B})} \leqslant C_{p,s_0} \delta^p \|u\|_{-s_0} \tag{5}$$

$$\begin{aligned}\|Q_0^\nu \tilde{\Gamma}_\delta f_\nu\|_{L^\infty(\mathscr{B})} &\leqslant (\gamma(x_0, \delta)^{-2} + |\lambda|)^{1-s/2} \, |||Q_0^\nu \tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-2} \\ &\leqslant C(\gamma(x_0, \delta)^{-2} + |\lambda|)^{1-s/2} \, |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-2}\end{aligned} \tag{6}$$

$$\|Q^\nu_{\text{error}} f_\nu\|_{L^\infty(\mathscr{B})} \leqslant C_{p,s_0} \delta^p \|f_\nu\|_{-s_0}. \tag{7}$$

To understand $S\Gamma_\delta u$, recall that $S = \lambda(x) T$ where $|\lambda(x)| \leqslant C\,\delta\gamma(x_0, \delta)^{-2}$ on $\mathscr{B}$, and T is a first-order pseudodifferential operator. Thus, since Γ_δ has symbol supported in $\{|\xi| \sim M/\delta\}$, we have

$$S\Gamma_\delta u = \lambda(x)\, T\Gamma_\delta u = \lambda(x) \cdot \frac{M}{\delta} Q'' \Gamma_\delta u + Q''_{\text{error}},$$

where Q'' is a symbol of order zero supported in $\{|\xi| \sim M/\delta\}$ and $\|Q''_{\text{error}} u\|_{L^\infty} \leqslant C_{p,s_0} \delta^p \|u\|_{-s_0}$. Hence

$$\begin{aligned}\|S\Gamma_\delta u\|_{L^\infty(\mathscr{B})} &\leqslant \left\| \frac{M}{\delta} \lambda(x) \right\|_{L^\infty(\mathscr{B})} \cdot \|Q'' \Gamma_\delta u\|_{L^\infty(\mathscr{B})} + \|Q''_{\text{error}} u\|_{L^\infty(\mathscr{B})} \\ &\leqslant C\gamma^{-2}(x_0, \delta) \, \|Q'' \Gamma_\delta u\|_{L^\infty(\mathscr{B})} + C_{p,s_0} \delta^p \|u\|_{-s_0} \\ &\leqslant C\gamma^{-2}(x_0, \delta)(\gamma(x_0, \delta)^{-2} + |\lambda|)^{-s/2} \\ &\quad \times |||Q'' \Gamma_\delta u|||_{\delta,\lambda,s} + C_{p,s_0} \delta^p \|u\|_{-s_0} \\ &\leqslant C\gamma^{-2}(x_0, \delta)(\gamma(x_0, \delta)^{-2} + |\lambda|)^{-s/2} \\ &\quad \times |||\Gamma_\delta u|||_{\delta,\lambda,s} + C_{p,s_0} \delta^p \|u\|_{-s_0}.\end{aligned} \tag{8}$$

Putting estimates (3)–(8) into the definition of F_0, we find that

$$\frac{\|F_0\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1-s/2}} \leqslant C\,|||\Gamma_\delta f_0|||_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-2}$$
$$+ C_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_{\nu\geqslant 1}\|f_\nu\|_{-s_0}\right)$$
$$+\frac{C\gamma(x_0,\delta)^{-2}}{(\gamma(x_0,\delta)^{-2}+|\lambda|)}\,|||\Gamma_\delta u|||_{\delta,\lambda,s}$$
$$+ C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-2}. \tag{9}$$

Regarding the $F_\nu(\nu\geqslant 1)$, we have either $F_\nu = \Gamma_\delta f_\nu$, in which case

$$\|F_\nu\|_{L^\infty(\mathscr{B})} \leqslant (\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}\,|||\Gamma_\delta f_\nu|||_{\delta,\lambda,s-1}; \tag{10A}$$

or else $F_\nu = Q_j\tilde{\Gamma}_\delta u$, in which case

$$\|F_\nu\|_{L^\infty(\mathscr{B})} \leqslant (\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}\,|||Q_j\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}$$
$$\leqslant C(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}. \tag{10B}$$

Consequently,

$$\frac{\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}} \leqslant C\sum_{\nu\geqslant 1} |||\Gamma_\delta f_\nu|||_{\delta,\lambda,s-1} + C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}. \tag{10}$$

We shall need also our basic localized L^2-estimate, which asserts that

$$\|\Gamma_\delta u\|_{\delta,\lambda,s} \leqslant C\,\|\tilde{\Gamma}_\delta f_0\|_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1}\|\tilde{\Gamma}_\delta f_\nu\|_{\delta,\lambda,s-1}$$
$$+ C_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_{\nu\geqslant 1}\|f_\nu\|_{-s_0}\right).$$

Since $\|g\|_{\delta,\lambda,s} \leqslant |||g|||_{\delta,\lambda,s}$, we obtain from the definition of $\|\Gamma_\delta u\|_{\delta,\lambda,s}$ the L^2-estimate

$$\frac{\|\Gamma_\delta u\|_{L^2(\mathscr{B})}}{(\gamma^{-2}(x_0,\delta)+|\lambda|)^{-s/2}\sqrt{\operatorname{vol}\mathscr{B}}} \leqslant C\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1}|||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1}$$
$$+ C\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|\right). \tag{11}$$

Now we know that $(L-\lambda)(\Gamma_\delta u) = F_0 + \sum_{\nu\geqslant 1}\mathscr{W}_\nu F_\nu$ with $\mathscr{W}_\nu$ smooth and subunit, and F_0, F_ν, $\Gamma_\delta u$ estimated by (9), (10), (11). We are in a position to apply our PDE estimates, Lemmas D′ and E′ in Section 5, with

$\gamma = \gamma(x_0, \delta)$. If $|\lambda| \geqslant B^2\gamma^{-2}(x_0\,\delta)$ then we apply Lemma E′. If $|\lambda| < B^2\gamma^{-2}(x_0, \delta)$, then we rewrite our equation in the form

$$L(\Gamma_\delta u) = \{F_0 + \lambda\Gamma_\delta u\} + \sum_{\nu \geqslant 1} \mathscr{W}_\nu F_\nu$$

and apply Lemma D′. This shows that if $|\lambda| \geqslant B^2\gamma^{-2}(x_0, \delta)$ then

$$|\Gamma_\delta u(x_0)| \leqslant \frac{C}{|\lambda|}\,\|F_0\|_{L^\infty(\mathscr{B})} + \frac{C}{|\lambda|^{1/2}} \sum_{\nu \geqslant 1} \|F_\nu\|_{L^\infty(\mathscr{B})} + \frac{C}{B}\,\|\Gamma_\delta u\|_{L^\infty(\mathscr{B})}. \tag{12}$$

If $|\lambda| \leqslant B^2\gamma^{-2}(x_0, \delta)$ then

$$\begin{aligned} |\Gamma_\delta u(x_0)| \leqslant{}& a\cdot(\gamma(x_0, \delta)^2\,\|F_0\|_{L^\infty(\mathscr{B})} + B^2\,\|\Gamma_\delta u\|_{L^\infty(\mathscr{B})}) \\ &+ C \sum_{\nu \geqslant 1} \gamma(x_0, \delta)\,\|F_\nu\|_{L^\infty(\mathscr{B})} \\ &+ C(a)\,\frac{\|\Gamma_\delta u\|_{L^2(\mathscr{B})}}{\sqrt{\operatorname{vol}\mathscr{B}}} \qquad \text{for} \quad 0 < a < 1. \end{aligned} \tag{13}$$

Again, B is a large constant to be determined later. Recall (9), (10), (11), which bound all the quantities on the right in (12), (13) except for $\|\Gamma_\delta u\|_{L^\infty(\mathscr{B})}$, which we bound simply by $(\gamma^{-2}(x_0, \delta) + |\lambda|)^{-s/2}\,|||\Gamma_\delta u|||_{\delta,\lambda,s}$. Therefore, we know the following estimates.

If $|\lambda| \geqslant B^2\gamma^{-2}(x_0, \delta)$, then

$$\begin{aligned} \frac{|\Gamma_\delta u(x_0)|}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{-s/2}} \leqslant{}& C\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C \sum_{\nu \geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} \\ &+ C_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\right) \\ &+ \frac{C}{B}\,|||\Gamma_\delta u|||_{\delta,\lambda,s} + C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}. \end{aligned}$$

If $|\lambda| \leqslant B^2\gamma^{-2}(x_0, \delta)$, then

$$\begin{aligned} \frac{|\Gamma_\delta u(x_0)|}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{-s/2}} \leqslant{}& C(a, B)\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C(a, B) \\ &\times \sum_{\nu \geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} + C(a, B, p, s_0) \\ &\times \delta^p\left(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\right) \\ &+ a\cdot(C + B^2)\,|||\Gamma_\delta u|||_{\delta,\lambda,s} \\ &+ C(a, B)\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}. \end{aligned}$$

In either case we have

$$\frac{|\Gamma_\delta u(x_0)|}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{-s/2}} \leqslant C(a,B)\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2}$$

$$+ C(a,B) \sum_{\nu\geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1}$$

$$+ C(a,B,p,s_0)\left(\|u\|_{-s_0} + \sum_{\nu\geqslant 1} \|f_\nu\|_{-s_0}\right)$$

$$+ \left[\frac{C}{B} + a\cdot(C+B^2)\right] |||\Gamma_\delta u|||_{\delta,\lambda,s}$$

$$+ C(a,B)\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}.$$

Taking the supermum over all x_0 on the left-hand side, we see that

$$|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C(a,B)\cdot |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C(a,B) \sum_{\nu\geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1}$$

$$+ C(p,s_0)\left(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\right) + |||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}$$

$$+ \left[\frac{C}{B} + a\cdot(C+B^2)\right] |||\Gamma_\delta u|||_{\delta,\lambda,s}. \tag{14}$$

Now we are ready to pick the constants a and B once and for all. We first pick B large enough and then pick a small enough, so that $[C/B + a\cdot(C+B^2)] < 1/2$. The last term on the right in (14) can therefore be absorbed into the left-hand side. From now on, we stop keeping track of the dependence of the constants in our estimates on a, B. Hence, (14) becomes

$$|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + \sum_{\nu\geqslant 1} C\,|||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1}$$

$$+ C_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\right) + C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}. \tag{15}$$

The last term on the right is analogous to the left-hand side, but with s replaced by $s-1$. Hence we may feed (15) into itself in an obvious induction to obtain

$$|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C_k\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C_k \sum_{\nu\geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1}$$

$$+ C^k_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\right)$$

$$+ C_k\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-k} \quad \text{for} \quad k = 1, 2, ... \tag{16}$$

If k is large enough, then $|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-k}$ is dominated by $C^k_{p,s_0}\,\delta^p\,\|u\|_{-s_0}$ and therefore (16) takes the form

$$|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1}|||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} + C\,\delta^p\Big(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\Big). \tag{17}$$

Equation (17) is our basic L^∞ estimate for solutions of $(A-\lambda)u = f_0 + \sum_\nu \varkappa_\nu f_\nu$.

Armed with (17), we now make a sharper examination of Eq. (2), by improving the key estimates (9) and (10). From (9) and (17) we get at once

$$\frac{\|F_0\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1-s/2}} \leqslant C\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1}|||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} + C\,\delta^p\Big(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\Big). \tag{18}$$

Here we need both (17) and its analogue with $\Gamma_\delta u$ replaced by $\tilde{\Gamma}_\delta u$.

Equation (10) was derived from (10A) and (10B). Leaving (10A) alone, we replace s by $s+1$ in (10B). In place of (10) we then obtain

$$\frac{\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}} \leqslant C\sum_{\nu\geqslant 1}|||\Gamma_\delta f_\nu|||_{\delta,\lambda,s-1} + C(\gamma(x_0,\delta)^{-2}+|\lambda|)^{-1/2}\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s}.$$

Using (17) and the estimate $C(\gamma(x_0,\delta)^{-2}+|\lambda|)^{-1/2} \leqslant C\gamma(x_0,\delta) \leqslant C\,\delta^\varepsilon$, we conclude that

$$\frac{\sum_{\nu\geqslant 1}\|F_\nu\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0,\ \delta)^{-2}+|\lambda|)^{1/2-s/2}} \leqslant C\,\delta^\varepsilon\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1}|||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} + C\,\delta^p\Big(\|u\|_{-s_0} + \sum_\nu \|f\|_{-s_0}\Big). \tag{19}$$

Immediately from (17) we get also

$$\frac{\|\Gamma_\delta u\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{-s/2}} \leqslant C\,|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\sum_{\nu\geqslant 1}|||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} + C\,\delta^p\Big(\|u\|_{-s_0} + \sum_\nu \|f_\nu\|_{-s_0}\Big). \tag{20}$$

In (18), (19), and (20) we have $\mathscr{B} = B_L(x_0, \gamma(x_0,\delta))$. We now shrink $\mathscr{B}$

to $\mathscr{B} = B_L(x_0, \gamma)$ with $\gamma = (\gamma(x_0, \delta)^{-2} + |\lambda|)^{-1/2} \leqslant \gamma(x_0, \delta)$. Estimates (18), (19), and (20) still hold a fortiori.

We are now in a position to apply Lemma B′ in Section 5 to Eq. (2), namely

$$L(\Gamma_\delta u) = \{F_0 + \lambda \Gamma_\delta u\} + \sum_{v \geqslant 1} \mathscr{W}_v F_v.$$

For η in Lemma B′ we take an arbitrarily high power of δ. The conclusion of Lemma B′, together with (18), (19), and (20) shows

$$\begin{aligned} \frac{|Y\Gamma_\delta u(x_0)|}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{1/2 - s/2}} \leqslant {} & C \, |||\tilde{\Gamma}_\delta f_0|||_{\delta, \lambda, s-2} + C_K \, |\ln \delta| \sum_{v \geqslant 1} |||\tilde{\Gamma}_\delta f_v|||_{\delta, \lambda, s-1} \\ & + C \, \delta^K \sum_{v \geqslant 1} \|\mathscr{W}_v F_v\|_{L^\infty(\mathscr{B})} \\ & + C \, \delta^p \left(\|u\|_{-s_0} + \sum_v \|f_v\|_{-s_0} \right), \end{aligned} \tag{21}$$

for subunit Y and for K an arbitarily large constant.

Recall that F_v denotes either $\Gamma_\delta f_v$ or $Q_j \tilde{\Gamma}_\delta u$ $(v \geqslant 1)$ with Q_j a pseudo-differential operator of order zero with symbol supported in $\{|\xi| \sim M/\delta\}$, Hence either

$$\|\mathscr{W}_v F_v\|_{L^\infty} = \|\mathscr{W}_v \Gamma_\delta f_v\|_{L^\infty} \leqslant C_{s_0} \, \delta^{-m(s_0)} \, \|f_v\|_{-s_0}$$

or else

$$\|\mathscr{W}_v F_v\|_{L^\infty} = \|\mathscr{W}_v Q_j \tilde{\Gamma}_\delta u\|_{L^\infty} \leqslant C_{s_0} \, \delta^{-m(s_0)} \, \|u\|_{-s_0}.$$

Taking K large enough $(K > p + m(s_0))$, we see that

$$C \, \delta^K \sum_{v \geqslant 1} \|\mathscr{W}_v F_v\|_{L^\infty(\mathscr{B})} \leqslant C' \, \delta^p \left(\|u\|_{-s_0} + \sum_v \|f_v\|_{-s_0} \right).$$

Hence (21) becomes

$$\begin{aligned} \frac{|Y\Gamma_\delta u(x_0)|}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{1/2 - s/2}} \leqslant {} & C \, |||\tilde{\Gamma}_\delta f_0|||_{\delta, \lambda, s-2} \\ & + C \, |\ln \delta| \sum_{v \geqslant 1} |||\tilde{\Gamma}_\delta f_v|||_{\delta, \lambda, s-1} \\ & + C \, \delta^p \left(\|u\|_{-s_0} + \sum_v \|f_v\|_{-s_0} \right). \end{aligned} \tag{22}$$

Taking the supremum over all x_0 on the left, we see that

$$\begin{aligned}|||Y\Gamma_\delta u|||_{\delta,\lambda,s-1} \leqslant C\ & |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2}\\ & + C\,|\ln\delta| \sum_{\nu\geqslant 1} |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1}\\ & + C\,\delta^p\Big(\|u\|_{-s_0}+\sum_\nu \|f_\nu\|_{-s_0}\Big).\end{aligned} \tag{23}$$

In (17) and (23), u is any solution of $(A-\lambda)u=f_0+\sum_{\nu\geqslant 1}\varkappa_\nu f_\nu$ with Y subunit and $\varkappa_\nu$ smooth, subunit.

Now suppose $f_\nu=0$ for $(\nu\geqslant 1)$. Then (17) and (23) become

$$|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C\ |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0}+\|f_0\|_{-s_0}) \tag{24}$$

$$|||Y\Gamma_\delta u|||_{\delta,\lambda,s-1} \leqslant C\ |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0}+\|f_0\|_{-s_0}). \tag{25}$$

Again recall that F_ν $(\nu\geqslant 1)$ denotes either $\Gamma_\delta f_\nu\equiv 0$, or else $Q_j\tilde{\Gamma}_\delta u$. For $F_\nu\not\equiv 0$ we have therefore

$$\mathscr{W}_\nu F_\nu=\mathscr{W}_\nu Q_j\tilde{\Gamma}_\delta u=Q_j\mathscr{W}_\nu\tilde{\Gamma}_\delta u+Q_0''\tilde{\Gamma}_\delta u+Q_{\text{error}}u \tag{26}$$

with Q_j, Q_0'' pseudodifferential operators of order zero with symbols supported in $\{|\xi|\sim M/\delta\}$ and $\|Q_{\text{error}}\,u\|_{L^\infty}\leqslant C\,\delta^p\,\|u\|_{-s_0}$.

Now

$$\begin{aligned}|||Q_j\mathscr{W}_\nu\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1} &\leqslant C\ |||\mathscr{W}_\nu\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}\\ &\qquad\text{(by (b) of Lemma 4 of Section 4)}\\ &\leqslant C\ |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2}\\ &\qquad + C\,\delta^p(\|u\|_{-s_0}+\|f_0\|_{-s_0})\qquad\text{by}\quad(25)\end{aligned}$$

with $Y=\mathscr{W}_\nu$ (and with $\tilde{\Gamma}_\delta$, $\tilde{\tilde{\Gamma}}_\delta$ in place of Γ_δ, $\tilde{\Gamma}_\delta$). Similarly,

$$\begin{aligned}|||Q_0''\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1} &\leqslant C\ |||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}\\ &\leqslant C\ |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2}+C\,\delta^p(\|u\|_{-s_0}+\|f_0\|_{-s_0}).\end{aligned}$$

Hence

$$\frac{|(Q_j\mathscr{W}_\nu\tilde{\Gamma}_\delta u+Q_0''\tilde{\Gamma}_\delta u)(x_0)|}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}}\leqslant C\ |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2}+C\,\delta^p(\|u\|_{-s_0}+\|f_0\|_{-s_0})$$

so that by (26),

$$\frac{|\mathscr{W}_\nu F_\nu(x_0)|}{(\gamma(x_0,\delta)^{-2}+|\lambda|)^{1/2-s/2}}\leqslant C\ |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2}+C\,\delta^p(\|u\|_{-s_0}+\|f_0\|_{-s_0}). \tag{27}$$

Since $f_\nu = 0$ ($\nu \geqslant 1$), Eq. (18) shows that

$$\frac{|F_0(x_0)|}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{1-s/2}} \leqslant C\, |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}) \tag{28}$$

while Eq. (20) yields

$$\frac{\|\Gamma_\delta u\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{-s/2}} \leqslant C\, |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}) \tag{29}$$

with $\mathscr{B} = B_L(x_0, \gamma)$, $\gamma = (\gamma(x_0, \delta)^{-2} + |\lambda|)^{-1/2}$.

Now $L(\Gamma_\delta u) = \lambda \Gamma_\delta u + F_0 + \sum_{\nu \geqslant 1} \mathscr{W}_\nu F_\nu \equiv F_0^\#$, and (27), (28), and (29) show that

$$\frac{\|F_0^\#\|_{L^\infty(\mathscr{B})}}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{1-s/2}} \leqslant C\, |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}) \tag{30}$$

with $\mathscr{B} = B_L(x_0, \gamma)$, $\gamma = (\gamma(x_0, \delta)^{-2} + |\lambda|)^{-1/2}$ again.

We are in position to apply Lemma C′ with η equal to an arbitrarily high power of δ. The conclusion of Lemma C′ and estimates (29), (30) show that for smooth, subunit Y, Y' we have

$$\frac{|YY'\Gamma_\delta u(x_0)|}{(\gamma(x_0, \delta)^{-2} + |\lambda|)^{1-s/2}} \leqslant C_K\, |\ln \delta|\; |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2}$$
$$+ C\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}) + C\,\delta^K \|\Gamma_\delta u\|_{C^3(\mathbb{R}^n)}. \tag{31}$$

Since $\|\Gamma_\delta u\|_{C^3(\mathbb{R}^n)} \leqslant C\,\delta^{-m(s_0)} \|u\|_{-s_0}$, the last term in (31) is dominated by $C\,\delta^p \|u\|_{-s_0}$ for K large. Hence the last term in (31) may be omitted. Now taking the supremum over all x_0 on the left of (31), we get

$$|||YY'\Gamma_\delta u|||_{\delta,\lambda,s-2} \leqslant C\,|\ln \delta|\; |||\tilde{\tilde{\Gamma}}_\delta f_0|||_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0}). \tag{32}$$

We change notation by taking $s+2$ in place of s, and $\tilde{\Gamma}_\delta$ in place of $\tilde{\tilde{\Gamma}}_\delta$. Hence (32) implies

$$|||YY'\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C\,|\ln \delta|\; |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s} + C\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}). \tag{33}$$

Estimate (33) holds for solutions of $(A - \lambda)u = f_0$.

We also want an L^∞-estimate for the equation

$$(A - \lambda)u = (S - \lambda)f. \tag{34}$$

As in the derivation of (2) we apply Γ_δ to both sides, and commute Γ_δ past both A and S. The result is

$$(A-\lambda)\,\Gamma_\delta u = (S-\lambda)\,\Gamma_\delta f + Q_0\tilde{\Gamma}_\delta u + Q_0'\tilde{\Gamma}_\delta f + \sum_{\nu\geqslant 1} Y_\nu Q_0^\nu \tilde{\Gamma}_\delta u + Q_{\text{error}}u + Q'_{\text{error}}f, \tag{35}$$

where Y_ν are smooth, subunit; Q_0, Q_0', Q_0^ν are pseudodifferential operators of order zero with symbols supported in $\{|\xi| \sim M/\delta\}$; and $\|Q_{\text{error}}\,g\|_{L^\infty}$, $\|Q'_{\text{error}}\,g\|_{L^\infty} \leqslant C\,\delta^p\,\|g\|_{L^\infty}$.

Once more we take $\mathscr{B} = B_L(x_0, \gamma(x_0, \delta))$. Estimate (8) above, applied to f, gives

$$\|S\Gamma_\delta f\|_{L^\infty(\mathscr{B})} \leqslant C\gamma^{-2}(x_0,\delta)(\gamma^{-2}(x_0,\delta)+|\lambda|)^{-s/2}\,|\!|\!|\Gamma_\delta f|\!|\!|_{\delta,\lambda,s} + C\,\delta^p\,\|f\|_{-s_0}. \tag{36}$$

By definition of $|\!|\!|\cdot|\!|\!|$ and by Lemma 4(b) in Section 4 (boundedness of pseudodifferential operator) we have

$$\|\lambda\Gamma_\delta f\|_{L^\infty(\mathscr{B})} \leqslant |\lambda|\,(\gamma^{-2}(x_0,\delta)+|\lambda|)^{-s/2}\,|\!|\!|\Gamma_\delta f|\!|\!|_{\delta,\lambda,s} \tag{37}$$

$$\begin{aligned}\|Q_0\tilde{\Gamma}_\delta u\|_{L^\infty(\mathscr{B})} &\leqslant (\gamma^{-2}(x_0,\delta)+|\lambda|)^{1-s/2}\,|\!|\!|Q_0\tilde{\Gamma}_\delta u|\!|\!|_{\delta,\lambda,s-2}\\ &\leqslant C(\gamma^{-2}(x_0,\delta)+|\lambda|)^{1-s/2}\,|\!|\!|\tilde{\Gamma}_\delta u|\!|\!|_{\delta,\lambda,s-2}\end{aligned} \tag{38}$$

$$\begin{aligned}\|Q_0'\tilde{\Gamma}_\delta f\|_{L^\infty(\mathscr{B})} &\leqslant (\gamma(x_0,\delta)^{-2}+|\lambda|)^{1-s/2}\,|\!|\!|Q_0'\tilde{\Gamma}_\delta f|\!|\!|_{\delta,\lambda,s-2}\\ &\leqslant C(\gamma,(x_0,\delta)^{-2}+|\lambda|)^{1-s/2}\,|\!|\!|\tilde{\Gamma}_\delta f|\!|\!|_{\delta,\lambda,s-2}\end{aligned} \tag{39}$$

$$\|Q_{\text{error}}u + Q'_{\text{error}}f\|_{L^\infty(\mathscr{B})} \leqslant C\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0}). \tag{40}$$

Together, (36)–(40) show that

$$F_0^+ = (S-\lambda)\,\Gamma_\delta f + Q_0\tilde{\Gamma}_\delta u + Q_0'\tilde{\Gamma}_\delta f + Q_{\text{error}}\,u + Q'_{\text{error}}f \tag{41}$$

satisfies

$$\|F_0^+\|_{L^\infty(\mathscr{B})} \leqslant C(\gamma^{-2}(x_0,\delta)+|\lambda|)^{1-s/2}\,\{|\!|\!|\tilde{\Gamma}_\delta f|\!|\!|_{\delta,\lambda,s} + |\!|\!|\tilde{\Gamma}_\delta u|\!|\!|_{\delta,\lambda,s-2} + C\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0})\},$$

i.e.,

$$|||F_0^+|||_{\delta,\lambda,s-2} \leqslant C\,|||\tilde{\Gamma}_\delta f|||_{\delta,\lambda,s} + C\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0}) + C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-2}. \tag{42}$$

Also, $F_\nu^+ \equiv Q_0^\nu \tilde{\Gamma}_\delta u$ satisfies

$$|||F_\nu^+|||_{\delta,\lambda,s-1} \leqslant C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1} \tag{43}$$

by Lemma 4(b) of Section 4 (boundedness of pseudodifferential operator).

Comparing (35) with the definitions of F_0^+, F_ν^+, we see that

$$(A-\lambda)(\Gamma_\delta u) = F_0^+ + \sum_{\nu \geqslant 1} Y_\nu F_\nu^+.$$

Hence by estimate (17) above,

$$\begin{aligned} |||\Gamma_\delta u|||_{\delta,\lambda,s} &\leqslant C\,|||\tilde{\Gamma}_\delta F_0^+|||_{\delta,\lambda,s-2} + C \sum_{\nu \geqslant 1} |||\tilde{\Gamma}_\delta F_\nu^+|||_{\delta,\lambda,s-1} \\ &\quad + C\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|F_\nu\|_{-s_0}\right) \\ &\leqslant C\,|||F_0^+|||_{\delta,\lambda,s-2} + C \sum_{\nu \geqslant 1} |||F_\nu^+|||_{\delta,\lambda,s-1} \\ &\quad + C\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|F_\nu^+\|_{-s_0}\right) \end{aligned} \tag{44}$$

(again we use Lemma 4(b) of Section 4).

From the definitions of F_ν^+, we have

$$C\,\delta^p\left(\|u\|_{-s_0} + \sum_\nu \|F_\nu^+\|_{-s_0}\right) \leqslant C'\,\delta^p(\|u\|_{2-s_0} + \|f\|_{2-s_0}). \tag{45}$$

Putting (42), (43), (45) into (44) yields the estimate

$$\begin{aligned} |||\Gamma_\delta u|||_{\delta,\lambda,s} &\leqslant C\,|||\tilde{\Gamma}_\delta f|||_{\delta,\lambda,s} + C\,\delta^p(\|u\|_{2-s_0} + \|f\|_{2-s_0}) \\ &\quad + C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}. \end{aligned} \tag{46}$$

The last term in (46) is analogous to the left-hand side, but with s improved to $s-1$. An obvious induction lets us feed (46) into itself, to replace $C\,|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1}$ by $C_k\,|||\tilde{\Gamma}_\delta u|||_{s-k}$. For large k, this term is dominated by $C\,\delta^p\,\|u\|_{2-s_0}$ and may therefore be omitted in (46).

Writing s_0+2 in place of s_0 in (46), we therefore obtain

$$|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C\, |||\tilde{\Gamma}_\delta f|||_{\delta,\lambda,s} + C\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0}) \tag{47}$$

for $(A-\lambda)u=(S-\lambda)f$.

Next we commute Γ_δ to the outside in (23) and (33). If $(A-\lambda)u = f_0+\sum_{v\geqslant 1} \varkappa_v f_v$ with $\varkappa_v$ smooth, subunit, and if Y, Y' are smooth and subunit, then

$$|||\Gamma_\delta Yu|||_{\delta,\lambda,s-1} \leqslant C\, |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} + C\,|\ln\delta| \sum_{v\geqslant 1} |||\tilde{\Gamma}_\delta f_v|||_{\delta,\lambda,s-1}$$
$$+ C\,\delta^p\left(\|u\|_{-s_0} + \sum_v \|f_v\|_{-s_0}\right) \tag{48}$$

in general; and

$$|||\Gamma_\delta YY'u|||_{\delta,\lambda,s} \leqslant C\,|\ln\delta|\; |||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s} + C\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}) \tag{49}$$

in case $f_v\equiv 0$ for $v\geqslant 1$.

These estimates are trivial consequences of (17), (23), and (33) since

$$\Gamma_\delta Yu = Y\Gamma_\delta u + Q_0\tilde{\Gamma}_\delta u + Q_{\text{error}}u$$

and

$$\Gamma_\delta YY'u = YY'\Gamma_\delta u + Q_0' Y\tilde{\Gamma}_\delta u + Q'' Y'\tilde{\Gamma}_\delta u + Q_0'''\tilde{\Gamma}_\delta u + Q_{\text{error}}' u$$

with Q_0, Q_0', Q_0'', Q_0''' pseudodifferential operator of order zero supported in $\{|\xi|\sim M/\delta\}$ and $\|Q'_{\text{error}}u\|_{L^\infty} + \|Q''_{\text{error}}u\|_{L^\infty} \leqslant C\,\delta^p\,\|u\|_{-s_0}$.

Main Lemma. *Say $|\lambda|\leqslant C$ or else $|\operatorname{Im}\lambda|\geqslant c\,|\lambda|$.*

(a) *Suppose $(A-\lambda)u=f_0+\sum_{v\geqslant 1}\varkappa_v f_v$ with $\varkappa_v$ smooth, subunit. Then*

$$\|\Gamma_\delta u\|_{L^\infty} \leqslant \frac{C}{|\lambda|}\,\|\tilde{\Gamma}_\delta f_0\|_{L^\infty} + \sum_{v\geqslant 1}\frac{C}{|\lambda|^{1/2}}\,\|\tilde{\Gamma}_\delta f_v\|_{L^\infty}$$
$$+ C_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_v \|f_v\|_{-s_0}\right);$$

and

$$\|\Gamma_\delta Yu\|_{L^\infty} \leqslant \frac{C}{|\lambda|^{1/2}}\,\|\tilde{\Gamma}_\delta f_0\|_{L^\infty} + C\,|\ln\delta| \sum_{v\geqslant 1}\|\tilde{\Gamma}_\delta f_v\|_{L^\infty}$$
$$+ C_{p,s_0}\,\delta^p\left(\|u\|_{-s_0} + \sum_v \|f_v\|_{-s_0}\right)$$

for smooth, subunit Y.

(b) *Suppose* $(A-\lambda)u=f_0$. *Then for smooth, subunit* Y *and* Y' *we have*

$$\|\Gamma_\delta YY'u\|_{L^\infty} \leqslant C\,|\ln\delta|\,\|\tilde{\Gamma}_\delta f_0\|_{L^\infty} + C_{p,s_0}\,\delta^p(\|u\|_{-s_0} + \|f_0\|_{-s_0}).$$

(c) *Suppose* $(A-\lambda)u=(S-\lambda)f$. *Then*

$$\|\Gamma_\delta u\|_{L^\infty} \leqslant C\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C_{p,s_0}\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0}).$$

Proof. Take $s=1$ in (17), and the first inequality in (a) is immediate. Take $s=1$ in (48), and the second inequality in (a) is immediate. Take $s=0$ in (49), and (b) is immediate. Take $s=0$ in (47), and (c) is immediate. (To derive (a) one uses $|\!|\!|w|\!|\!|_{\delta,\lambda,-1} \leqslant (C/|\lambda|^{1/2})\,\|w\|_{L^\infty}$.) ∎

10. Global Estimates

Suppose now we are on a compact manifold without boundary. Let A be the pseudodifferential operator which we have so far studied locally, and let λ be a complex number lying on the contour $\mathscr{C}_B = \{\operatorname{Re}\lambda = -B, |\operatorname{Im}\lambda| < B\} \cup \{\operatorname{Re}\lambda \geqslant -B, |\operatorname{Im}\lambda| = \operatorname{Re}\lambda + 2B\}$. Our goal is to understand the global equation $(A-\lambda)u=f$. Results follow easily from our Main Lemma, once we have proved the following

Lemma. *If* B *is large enough, depending on* s_0, *then we have*

$$\|(A-\lambda)u\|_{-s_0} \geqslant c\,|\lambda|\,\|u\|_{-s_0} \qquad \text{for} \quad \lambda\in\mathscr{C}_B.$$

Proof. Let Λ_{s_0} be an elliptic pseudodifferential operator of order s_0 with a global pseudodifferential operator inverse $\Lambda_{s_0}^{-1}$. Then $\Lambda_{s_0}\colon L^2 \to H^{-s_0}$ is an isomorphism, so our estimate amounts to the L^2-estimate

$$\|\Lambda_{s_0}^{-1}(A-\lambda)\,\Lambda_{s_0}w\| \geqslant c\,|\lambda|\,\|w\|. \tag{1}$$

Now $A=L+S$ with $L=\sum_j X_j^*\phi_j X_j$, $\phi_j \geqslant 0$. As in the deduction of (2) from (1) in Section 9 on L^∞ Estimates, we find that

$$\Lambda_{s_0}^{-1}[A, \Lambda_{s_0}] = \sum_{\nu\geqslant 1} Q_\nu Y_\nu + Q_0$$

with Q_0, Q_ν pseudodifferential operator of order zero, and Y_ν smooth, subunit vector fields. Thus

$$\|\Lambda_{s0}^{-1}(A-\lambda)\Lambda_{s0}w\| = \left\|(A-\lambda)w + \sum_{v\geqslant 1} Q_v Y_v w + Q_0 w\right\|$$

$$\geqslant \|(A-\lambda)w\| - C\sum_{v\geqslant 1}\|Y_v w\| - C\|w\| \tag{2}$$

since Q_0, Q_v are bounded on L^2.

As before, an elementary integration by parts shows that $\|Y_v w\|^2 \leqslant \langle Lw, w\rangle$ since Y_v is subunit. Note that we are integrating by parts on a compact manifold without boundary, so there are no troublesome boundary terms. Also, $0 \leqslant \langle Sw, w\rangle + C\|w\|^2$ by Garding's inequality. Hence

$$\begin{aligned}\|Y_v w\|^2 &\leqslant \langle (L+S)w, w\rangle + C\|w\|^2\\ &\leqslant \operatorname{Re}\langle (A-\lambda)w, w\rangle + (|\lambda| + C)\|w\|^2\\ &\leqslant a^2\|(A-\lambda)w\|^2 + (C(a) + |\lambda|)\|w\|^2\\ &\qquad \text{with}\quad 0<a\leqslant 1 \text{ to be determined.}\end{aligned}$$

Thus

$$C\sum_{v\geqslant 1}\|Y_v w\| + C\|w\| \leqslant Ca\|(A-\lambda)w\| + (C'(a) + C|\lambda|)^{1/2}\|w\|,$$

so that by (2) we get

$$\begin{aligned}\|\Lambda_{s0}^{-1}(A-\lambda)\Lambda_{s0}w\| &\geqslant (1-Ca)\|(A-\lambda)w\| - (C'(a) + C|\lambda|)^{1/2}\|w\|\\ &\geqslant \tfrac{1}{2}\|(A-\lambda)w\| - (C' + C|\lambda|)^{1/2}\|w\|\\ &\qquad \text{if}\quad \text{we pick } a < \frac{1}{2C}.\end{aligned} \tag{3}$$

On the other hand, for $\lambda \in \mathscr{C}_B$ we have either $|\operatorname{Im}\lambda| > c|\lambda|$, so that

$$\|\operatorname{Im}\langle (A-\lambda)w, w\rangle| = |\operatorname{Im}\lambda|\,\|w\|^2 \geqslant c|\lambda|\,\|w\|^2,$$

or else $\operatorname{Re}\lambda = -B$ so that

$$\operatorname{Re}\langle (A-\lambda)w, w\rangle = \operatorname{Re}\langle (A+B)w, w\rangle \geqslant (B-C)\|w\|^2 \geqslant c|\lambda|\,\|w\|^2$$

if B is picked large enough. In either case we get

$$c|\lambda|\,\|w\|^2 \leqslant |\langle (A-\lambda)w, w\rangle| \leqslant \|(A-\lambda)w\|\cdot\|w\|,$$

so that $\|(A-\lambda)w\| \geqslant c|\lambda|\,\|w\|$. Substituting this into (3), we get

$$\|\Lambda_{s0}^{-1}(A-\lambda)\Lambda_{s0}w\| \geqslant \{c|\lambda| - (C' + C|\lambda|)^{1/2}\}\,\|w\|.$$

If $\lambda \in \mathscr{C}_B$ and B is large enough, then the expression in curly brackets is larger than $c' |\lambda|$, and hence we obtain the desired estimate (1). ∎

COROLLARY 1. *If* $f \in H^{-s_0}$ *and* $\lambda \in \mathscr{C}_B$ $(B \gg 1$ *depending on* $s_0)$ *then* $(A-\lambda)u = f$ *has a unique solution* $u = (A-\lambda)^{-1} f \in H^{-s_0}$ *and* $\|u\|_{-s_0} \leqslant (C/|\lambda|)\, \|f\|_{-s_0}$.

Proof. Apply the lemma with $\bar{\lambda}$ in palce of λ, and with $-s_0$ in place of s_0. Thus

$$\|(A-\lambda)^* w\|_{s_0} \geqslant c\, |\lambda|\, \|w\|_{s_0},$$

which shows that $\{(A-\lambda)f \mid f \in C^\infty\}$ is dense in H^{-s_0}. The lemma shows that $(A-\lambda)^{-1}$ is a densely defined, bounded operator on H^{-s_0}. ∎

COROLLARY 2. *If* $B \gg 1$ *depending on* s_0, *then* $(L+B)u = f$ *has a unique solution* $u \in H^{-s_0}$, *for* $f \in H^{-s_0}$. *Moreover,* $\|u\|_{-s_0} \leqslant C\, \|f\|_{-s_0}$.

Proof. This is merely the special case $A = L$, $\lambda = -B \in \mathscr{C}_B$ of Corollary 1. ∎

When we define, say, $u = (A-\lambda)^{-1} \varkappa f$, then we obtain from the above lemma that $\|u\|_{-s_0} \leqslant C\, \|\varkappa f\|_{-s_0} \leqslant C\, \|f\|_{1-s_0}$ when $\lambda \in \mathscr{C}_B$. Hence in the Main Lemma from Section 9 on L^∞-estimates, the "junk" terms $\delta^p\, \|u\|_{-s_0}$ may be absorbed in the other "junk" terms $\sum_\nu \|f_\nu\|_{-s_0}$ after at worst changing s_0 by 1 or 2. Since s_0 is arbitrarily large anyway, this does not matter. Thus we may read off the following consequences of the Main Lemma.

GLOBAL MAIN LEMMA. *Suppose* $\lambda \in \mathscr{C}_B$ *with* $B \gg 1$ *depending on* s_0. *Let* Y, Y' *be smooth, subunit vector fields. Then*

(a) $\|\Gamma_\delta (A-\lambda)^{-1} f\|_{L^\infty} \leqslant (C/|\lambda|)\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}$

(b) $\|\Gamma_\delta Y (A-\lambda)^{-1} f\|_{L^\infty} \leqslant (C/|\lambda|^{1/2})\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}$

(c) $\|\Gamma_\delta (A-\lambda)^{-1}\, Y f\|_{L^\infty} \leqslant (C/|\lambda|^{1/2})\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}$

(d) $\|\Gamma_\delta Y Y' (A-\lambda)^{-1} f\|_{L^\infty} \leqslant C\, |\ln \delta|\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}$

(e) $\|\Gamma_\delta Y (A-\lambda)^{-1}\, Y'\|_{L^\infty} \leqslant C\, |\ln \delta|\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}$

(f) $\|\Gamma_\delta (A-\lambda)^{-1} (S-\lambda) f\|_{L^\infty} \leqslant C\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}$.

Proof. Immediate from the Main Lemma, by virtue of the preceding remarks. ∎

Of course (f) above is a technical tool to allow us to control $(A-\lambda)^{-1}\, YY'$. To carry this out, we need the next two elementary results.

LEMMA. *Suppose $\mathscr{L}$ is a self-adjoint second-order pseudodifferential operator satisfying the subelliptic estimate $\langle \mathscr{L}w, w\rangle \geqslant c\,\|w\|_\varepsilon^2$, and let $\mathscr{S}$ be any pseudodifferential operator. Let P, Q be self-adjoint pseudodifferential operators of order zero, whose symbols (also called P, Q) satisfy* $\operatorname{supp} P \subset \{|\xi| \sim \delta^{-1}\}$ *and distance* $(\operatorname{supp} P, \operatorname{supp} Q) \geqslant c\delta^{-1}$. *If $\mathscr{L}u = \mathscr{S}Pf$, then* $\|Qu\| \leqslant C\,\delta^p(\|u\|_{-s_0} + \|f\|_{-s_0})$.

Proof. (This is a standard fact about subelliptic equations.) We have

$$\|Q\mathscr{L}u\|_{s_0} = \|Q\mathscr{S}Pf\|_{s_0} = \|\Lambda^{s_0}Q\mathscr{S}Pf\| \leqslant C\,\delta^p\,\|f\|_{-s_0}$$

because of the disjoint supports of Q, P. Therefore

$$\begin{aligned} |\operatorname{Re}\langle Q^*Q\mathscr{L}u, u\rangle| &\leqslant C\,\|Q^*Q\mathscr{L}u\|_{s_0}\,\|u\|_{-s_0} \\ &\leqslant C\,\|Q\mathscr{L}u\|_{s_0}\,\|u\|_{-s_0} \\ &\leqslant C\,\delta^p(\|u\|^2_{-s_0} + \|f\|^2_{-s_0}). \end{aligned} \tag{4}$$

On the other hand, let $\tilde{Q}$ be another pseudodifferential operator of order zero, whose symbol (also called $\tilde{Q}$) satisfies $\operatorname{supp} \tilde{Q} \subset \{|\xi| \sim \delta^{-1}\}$, distance $(\operatorname{supp} P, \operatorname{supp} \tilde{Q}) > c'\,\delta^{-1}$, distance $(\operatorname{supp} Q, \operatorname{supp}(1-\tilde{Q})) > c''\delta^{-1}$. Then

$$\operatorname{Re}\langle Q^*Q\mathscr{L}u, u\rangle = O(\delta^p\,\|u\|^2_{-s_0}) + \operatorname{Re}\langle Q\mathscr{L}(\tilde{Q}u), Q(\tilde{Q}u)\rangle \tag{5}$$

and

$$\operatorname{Re}\langle Q\mathscr{L}w, Qw\rangle = \operatorname{Re}\langle \mathscr{L}Qw, Qw\rangle + O(\|w\|^2)$$

since $Q[Q, \mathscr{L}]$ is first-order with purely imaginary principal symbol; and thus

$$\begin{aligned} \operatorname{Re}\langle Q\mathscr{L}(\tilde{Q}u), Q(\tilde{Q}u)\rangle &= \langle \mathscr{L}Q\tilde{Q}u, Q\tilde{Q}u\rangle + O(\|\tilde{Q}u\|^2) \\ &= \langle \mathscr{L}Qu, Qu\rangle + O(\|\tilde{Q}u\|^2 + C\,\delta^p\,\|u\|^2_{-s_0}) \\ &\geqslant c\,\|Qu\|^2_\varepsilon - C\,\|\tilde{Q}u\|^2 - C\,\delta^p\,\|u\|^2_{-s_0} \\ &\geqslant c\,\delta^{-2\varepsilon}\,\|Qu\|^2 - C\,\|\tilde{Q}u\|^2 - C'\,\delta^p\,\|u\|^2_{-s_0}, \end{aligned}$$

since Q is supported in $\{|\xi| \sim \delta^{-1}\}$.

Comparing this estimate with (4) and (5), we obtain

$$\|Qu\|^2 \leqslant C\,\delta^{2\varepsilon}\,\|\tilde{Q}u\|^2 + C\,\delta^p(\|u\|^2_{-s_0} + \|f\|^2_{-s_0}). \tag{6}$$

Now $\tilde{Q}$ has all the properties we assumed for Q. Hence an obvious induction, feeding (6) into itself, allows us to prove

$$\|Qu\|^2 \leqslant C_k\,\delta^{\varepsilon k}\,\|\tilde{Q}u\|^2 + C_k\,\delta^p(\|u\|^2_{-s_0} + \|f\|^2_{-s_0}) \qquad \text{for any } k. \tag{7}$$

Picking k large enough yields $\delta^{\varepsilon k} \|\tilde{Q}u\|^2 \leqslant C\,\delta^p \|u\|^2_{-s_0}$, since the symbol $\tilde{Q}$ is supported in $\{|\xi| \sim \delta^{-1}\}$. Hence (7) implies $\|Qu\|^2 \leqslant C\,\delta^p(\|u\|^2_{-s_0} + \|f\|^2_{-s_0})$, completing the proof of the Lemma. ▌

LEMMA. *For Y, Y' smooth, subunit vector fields and $B \gg 1$ depending on p, s_0, we have the estimate*

$$\|\Gamma_\delta(L+B)^{-1}\,YY'f\|_{L^\infty} \leqslant C\,|\ln \delta|\;\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}.$$

Proof. Set $u = (L+B)^{-1}\,YY'f$, $v = (L+B)^{-1}\,YY'\tilde{\Gamma}_\delta f$, where the symbol of $\tilde{\Gamma}_\delta$ is equal to one on the support of the symbol of Γ_δ. Taking $s_1 = s_0 + p + 2$, and using boundedness of $(L+B)^{-1}$ on H^{-s} $(s = s_0 + 2,\ s = s_1)$, we obtain

$$\|u\|_{-s_1} \leqslant C\,\|u\|_{-s_0-2} \leqslant C'\,\|YY'f\|_{-s_0-2} \leqslant C''\,\|f\|_{-s_0} \tag{8}$$

$$\begin{aligned} \|v\|_{-s_1} &\leqslant C\,\|YY'\tilde{\Gamma}_\delta f\|_{-s_1} = C\,\|YY'\tilde{\Gamma}_\delta f\|_{-s_0-p-2} \\ &\leqslant C'\,\|\tilde{\Gamma}_\delta f\|_{-s_0-p} \leqslant C''\,\delta^p\,\|f\|_{-s_0}. \end{aligned} \tag{9}$$

The preceding lemma with $\mathscr{L} = L + B$, $\mathscr{S} = YY'$, $P = I - \tilde{\Gamma}_\delta$, $Q = \Gamma_\delta$ implies

$$\|\Gamma_\delta u - \Gamma_\delta v\| \leqslant C\,\delta^{p_1}(\|u - v\|_{-s_1} + \|f\|_{-s_1}) \leqslant C\,\delta^{p_1}\,\|f\|_{-s_0},$$

with p_1 arbitrarily large. Hence

$$\|\Gamma_\delta u - \Gamma_\delta v\|_{s_0} \leqslant C\,\delta^{-s_0}\,\|\Gamma_\delta u - \Gamma_\delta v\| + C\,\delta^p\,\|u - v\|_{-s_1} \leqslant C'\,\delta^p\,\|f\|_{-s_1},$$

provided we take $p_1 \geqslant p + s_0$. Consequently,

$$\|\Gamma_\delta u - \Gamma_\delta v\|_{L^\infty} \leqslant C\,\delta^p\,\|f\|_{-s_0}, \tag{10}$$

since we may assume s_0 is greater than the Sobolev index.

On the other hand, Lemma A* from the section on PDE Lemmas applies to $(L+B)v = YY'(\tilde{\Gamma}_\delta f)$, with $F_0 = F_\nu = 0$, $F_* = \tilde{\Gamma}_\delta f$, $U = v$, $\eta = \delta^K$ ($K \gg 1$ to be picked), $s = s_1$. Thus we obtain

$$\begin{aligned} \|v\|_{L^\infty} &\leqslant C_K\,|\ln \delta|\;\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\|v\|_{-s_1} + C\,\delta^{2K}\,\|YY'\tilde{\Gamma}_\delta f\|_{L^\infty} \\ &\leqslant C_K\,|\ln \delta|\;\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0} \\ &\quad + C\,\delta^{2K}\,\|(YY'\tilde{\Gamma}_\delta \Lambda^{s_0})(\Lambda^{-s_0}f)\|_{L^\infty} \end{aligned} \tag{11}$$

by virtue of (10). Now $YY'\tilde{\Gamma}_\delta \Lambda^{s_0}\colon L^2 \to L^\infty$ is a bounded operator with norm at most $C_{s_0}\,\delta^{-m(s_0)}$. Therefore, (11) implies

$$\begin{aligned} \|v\|_{L^\infty} &\leqslant C_K\,|\ln \delta|\;\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0} + C\,\delta^{2K-m(s_0)}\,\|\Lambda^{-s_0}f\|_{L^2} \\ &= C_k\,|\ln \delta|\;\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0} + C_{s_0}\,\delta^{2K-m(s_0)}\,\|f\|_{-s_0}. \end{aligned}$$

Taking K large enough, depending on s_0 and p, we conclude that

$$\|v\|_{L^\infty} \leqslant C\,|\ln \delta|\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}.$$

This implies

$$\|\Gamma_\delta v\|_{L^\infty} \leqslant C\,|\ln \delta|\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0},$$

and therefore from (10) we get

$$\|\Gamma_\delta u\|_{L^\infty} \leqslant C\,|\ln \delta|\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0},$$

which is the conclusion of the Lemma. ∎

At last we are in a position to supply the missing part of the Global Main Lemma, namely,

(g) $\|\Gamma_\delta (A-\lambda)^{-1}\,YY'f\| \leqslant C\,|\ln \delta|\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}.$

This is immediate from the above lemma, the Global Main Lemma, and the factorization

$$\begin{aligned}(A-\lambda)^{-1}\,YY' &= [(A-\lambda)^{-1}\,(L+B)][L+B)^{-1}\,YY'] \\ &= [(A-\lambda)^{-1}\,((A-\lambda)-(S-\lambda)+B)][(L+B)^{-1}\,YY'] \\ &= [I-(A-\lambda)^{-1}\,(S-\lambda)+B(A-\lambda)^{-1}][(L+B)^{-1}\,YY'].\end{aligned}$$

Now, estimates (a)–(g) of the Global Main Lemma are useful for $|\lambda| < \delta^{-K}$ (any K), but if $|\lambda|$ is very large compared to δ^{-1} then the junk terms $C\,\delta^p\,\|f\|_{-s_0}$ are too big.

Hence we need a substitute for the Global Main Lemma for $|\lambda| > \delta^{-1}$ (say). Fortunately, this is trivial to prove.

LEMMA. *Let P and Q be pseudodifferential operators of order zero with symbol (P) supported in $|\xi| > R$ and symbol (Q) supported in $|\xi| < (1/2)R$. If $\lambda \in \mathscr{C}_B$ with $B \gg 1$ depending on p, s_0, then we have*

$$\|Q(A-\lambda)^{-1}\,Pf\|_{s_0} \leqslant CR^{-p}\,\|f\|_{-s_0}.$$

Proof. Using the partition of unity we can split Q into $Q = Q_{\text{inner}} + \sum_\nu Q_\nu$ with each Q_ν having symbol supported in $\{|\xi| \sim 2^\nu\}$ with $R^{1/100} < 2^\nu < R$, and with Q_{inner} having symbol supported in $\{|\xi| < R^{1/100}\}$. There are $O(\ln R)$ terms, so it is enough to deal with each term separately and add up the results. We have

$$\|Q_\nu(A-\lambda)^{-1}\,Pf\|_{s_0} \leqslant C_K(2^\nu)^{-K}\,\|f\|_{-s_0}$$

for any K, by virtue of the Lemma above. Taking K large enough ($K \geqslant 100p+1$) we get

$$\|Q_\nu(A-\lambda)^{-1} Pf\|_{s_0} \leqslant CR^{-1/100-p} \|f\|_{-s_0} \qquad \text{for} \quad R^{1/100} < 2^\nu < R.$$

To handle Q_{inner}, we write (with $m \geqslant 0$ to be picked)

$$Q_{\text{inner}}(A-\lambda)^{-1} P = [Q_{\text{inner}} \Lambda^m][\Lambda^{-m}(A-\lambda)^{-1} \Lambda^m][\Lambda^{-m}P].$$

We have

$$\begin{aligned} Q_{\text{inner}} \Lambda^m \colon H_{-s_0} &\to H_{+s_0} && \text{with norm } O((R^{1/100})^{m+2s_0}) \\ \Lambda^{-m}(A-\lambda)^{-1} \Lambda^m \colon H_{-s_0} &\to H_{-s_0} && \text{with norm independent of } R \\ & && (R \text{ appears nowhere in this operator}) \\ \Lambda^{-m}P \colon H_{-s_0} &\to H_{-s_0} && \text{with norm } O(R^{-m}). \end{aligned}$$

Hence

$$\|Q_{\text{inner}}(A-\lambda)^{-1} Pf\|_{s_0} \leqslant C_m R^{(m+2s_0)/100-m} \|f\|_{-s_0} \leqslant CR^{-p} \|f\|_{-s_0}$$

if m is big enough. Summing our estimates now yields the conclusion of the Lemma. ∎

Now let $R = |\lambda|^{1/20}$, say, and define symbols $\Omega^\#$, Ω, $\tilde{\Omega}$ all equal to 1 for $|\xi| < R/2$, all supported in $|\xi| \leqslant 2R$, and with $\Omega = 1$ on $\operatorname{supp} \Omega^\#$, $\tilde{\Omega} = 1$ on $\operatorname{supp} \Omega$. Let $\Omega^\#$, Ω, $\tilde{\Omega}$ also denote the corresponding pseudodifferential operator.

Our first goal is to construct a good approximate solution to $(\tilde{\Omega} A - \lambda) u = \Omega f$. The equation may be rewritten as

$$\left[1 - \left(\frac{R^2}{\lambda}\right)\left(\frac{\tilde{\Omega} A}{R^2}\right)\right] u = -\frac{1}{\lambda} \Omega f.$$

Since $|R^2/\lambda| \leqslant 1/|\lambda|^{9/10}$ and $(\tilde{\Omega} A/R^2)$ is a pseudodifferential operator of order zero, it is natural to use a truncated Neumann series and set

$$Q_0 = -\sum_{k=0}^{K} \left(\frac{R^2}{\lambda}\right)^k \left(\frac{\tilde{\Omega} A}{R^2}\right)^k$$

as operators. Thus Q_0 is a pseudodifferential operator of order zero and for K large enough we have

$$(\tilde{\Omega} A - \lambda)\left[\frac{Q_0 \Omega f}{\lambda}\right] = \Omega f - |\lambda|^{-p_1} Q_1 \Omega f$$

with Q_1 another pseudodifferential operator of order zero and p_1 arbitrarily large. We have

$$\begin{aligned}|\lambda|^{-p_1} \|Q_1 \Omega f\|_{s_0} &\leqslant C |\lambda|^{-p_1} R^{2s_0} \|f\|_{-s_0}\\ &\leqslant C |\lambda|^{-p_1+2s_0/20} \|f\|_{-s_0}\\ &\leqslant C |\lambda|^{-p} \|f\|_{-s_0}.\end{aligned}$$

Write Q_{error} for any operator satisfying $\|Q_{\text{error}} f\| \leqslant C |\lambda|^{-p} \|f\|_{-s_0}$, and we have

$$(\tilde{\Omega} A - \lambda)\left[\frac{Q_0 \Omega f}{\lambda}\right] = \Omega f - Q_{\text{error}} f.$$

Moreover, $(1-\tilde{\Omega}) A Q_0 \Omega$ has the form Q_{error}, since it contains the factors $(1-\tilde{\Omega})$ and Ω. Hence,

$$(A-\lambda)\left[\frac{Q_0 \Omega f}{\lambda}\right] = \Omega f - Q_{\text{error}} f.$$

Therefore

$$(A-\lambda)\left\{(A-\lambda)^{-1} f - (A-\lambda)^{-1} Q_{\text{error}} f - \frac{Q_0 \Omega f}{\lambda}\right\} = (1-\Omega) f.$$

Since $(A-\lambda)^{-1}$ is bounded on H^{s_0} for $\lambda \in \mathscr{C}_B$ ($B \gg 1$ depending on s_0), $(A-\lambda)^{-1} Q_{\text{error}}$ again has the form Q_{error}, so that

$$(A-\lambda)\left\{(A-\lambda)^{-1} f - \frac{Q_0 \Omega f}{\lambda} - Q_{\text{error}} f\right\} = (1-\Omega) f.$$

The preceding lemma with $P = 1-\Omega$ and $Q = \Omega^{\#}$ now yields

$$\begin{aligned}\Omega^{\#}(A-\lambda)^{-1} f &= \Omega^{\#} \frac{Q_0 \Omega f}{\lambda} + Q_{\text{error}} f\\ &= \frac{\Omega^{\#} Q_0 f}{\lambda} + Q_{\text{error}} f.\end{aligned}$$

Now let Q, Q' be one of

$$\Gamma_\delta, I; \qquad \Gamma_\delta Y, I; \Gamma_\delta, Y; \qquad \Gamma_\delta Y Y', I; \Gamma_\delta Y, Y'; \Gamma_\delta, YY'.$$

Here Y and Y' are smooth, subunit vector fields, and we suppose

$1/\delta < |\lambda|^{1/100}$. Composing the above equation with Q on the left and Q' on the right, we get

$$Q\Omega^{\#}(A-\lambda)^{-1}\,Q' = \frac{Q\Omega^{\#}Q_0Q'}{\lambda} + Q_{\text{error}}.$$

Also, in each case $Q(1-\Omega^{\#}) = \Gamma_\delta \cdot (\text{some pseudodifferential operator}) \cdot (1-\Omega^{\#}) = Q_{\text{error}}$. Again since $(A-\lambda)^{-1}$ is bounded on H^{-s_0} for $\lambda \in \mathscr{C}_B$ ($B \gg 1$ depending on s_0) it follows that $Q_{\text{error}}(A-\lambda)^{-1}\,Q'$ again has the form Q_{error}.

Hence $Q(A-\lambda)^{-1}\,Q' = QQ_0Q'/\lambda + Q_{\text{error}}$ for each for the Q, Q' above.

We have

$$\|(\Gamma_\delta)\,Q_0 f\|_{L^\infty} \leqslant C\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}$$

$$\|(\Gamma_\delta Y)\,Q_0 f\|_{L^\infty} \leqslant \frac{C}{\delta}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}$$

$$\|(\Gamma_\delta)\,Q_0(Y) f\|_{L^\infty} \leqslant \frac{C}{\delta}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}$$

$$\|(\Gamma_\delta YY')\,Q_0 f\|_{L^\infty} \leqslant \frac{C}{\delta^2}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}$$

$$\|(\Gamma_\delta Y)\,Q_0(Y') f\|_{L^\infty} \leqslant \frac{C}{\delta^2}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}$$

$$\|(\Gamma_\delta)\,Q_0(YY') f\|_{L^\infty} \leqslant \frac{C}{\delta^2}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}.$$

Putting these estimates (which are trivial special cases of our L^∞-boundedness lemma for pseudodifferential operators) into the above formula, we obtain

$$\begin{aligned}
&(\mathrm{a}^{\#}) \quad \|\Gamma_\delta(A-\lambda)^{-1} f\|_{L^\infty} \leqslant (C/|\lambda|)(\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})\\
&(\mathrm{b}^{\#}) \quad \|\Gamma_\delta Y(A-\lambda)^{-1} f\|_{L^\infty} \leqslant (C/|\lambda|)(\delta^{-1}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})\\
&(\mathrm{c}^{\#}) \quad \|\Gamma_\delta(A-\lambda)^{-1}\,Yf\|_{L^\infty} \leqslant (C/|\lambda|)(\delta^{-1}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})\\
&(\mathrm{d}^{\#}) \quad \|\Gamma_\delta YY'(A-\lambda)^{-1} f\|_{L^\infty} \leqslant (C/|\lambda|)(\delta^{-2}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})\\
&(\mathrm{e}^{\#}) \quad \|\Gamma_\delta Y(A-\lambda)^{-1}\,Y'f\|_{L^\infty} \leqslant (C/|\lambda|)(\delta^{-2}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0})\\
&(\mathrm{f}^{\#}) \quad \|\Gamma_\delta(A-\lambda)^{-1}\,YY'f\|_{L^\infty} \leqslant (C/|\lambda|)(\delta^{-2}\,\|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\,\|f\|_{-s_0}).
\end{aligned}$$

These equations hold for $\lambda \in \mathscr{C}_B$, $B \gg 1$ depending on p, s_0, and for $1/\delta < |\lambda|^{1/20}$, i.e., for $\lambda > \delta^{-20}$.

At last we can analyze

$$e^{-sA} = \frac{1}{2\pi i} \oint_{\mathscr{C}_B} e^{-s\lambda}(A-\lambda)^{-1}\, d\lambda \qquad \text{with} \quad B \gg 1.$$

If Q and Q' are any operators, then

$$\|\Gamma_\delta Q e^{-sA} Q' f\|_{L^\infty} \leqslant \frac{1}{2\pi} \int_{\mathscr{C}_B} e^{-s\operatorname{Re}(\lambda)} \|\Gamma_\delta Q(A-\lambda)^{-1} Q' f\|_{L^\infty} |d\lambda|.$$

We use this for $Q = I$, Y, YY' and $Q' = I$, Y', YY', making sure QQ' has order $\leqslant 2$. For $|\lambda| \leqslant \delta^{-20}$, we use the Global Main Lemma (a)–(g), while for $|\lambda| > \delta^{-20}$ we use $(\mathrm{a}^{\#})$–$(\mathrm{f}^{\#})$ to estimate $\|\Gamma_\delta Q(A-\lambda)^{-1} Q' f\|_{L^\infty}$.

Theorem 1.1 is now trivial.

11. A Generalization: $e^{-t\tilde{A}}$

Let the setting be as before. Suppose we are given a complex vector field V and a pseudodifferential operator of order zero $\tilde{Q}$ such that $\tilde{A} = A + V\tilde{Q}$ is self-adjoint and there exist c, C, K constants such that

$$c((K+A)u, u) \leqslant ((K+\tilde{A})u, u) \leqslant C((K+A)u, u).$$

Also assume η is such that

$$\delta^\eta \geqslant (\gamma(x, \delta))^{1/10}$$

for all $x \in \mathscr{M}$, and assume $\delta^\eta V$ is subunit for L.

Then Theorem 1.1 holds for $e^{-t\tilde{A}}$. We proceed to prove that.

Main Lemma for $\tilde{A}$. *The Main Lemma holds with A replaced by $\tilde{A}$.*

Proof. Assume

$$(\tilde{A}-\lambda)u = f_0 + \sum_{v \geqslant 1} Z_v f_v$$

or

$$(A-\lambda)u = f_0 + \sum_{v \geqslant 1} Z_v f_v - \delta^{-\eta}\delta^\eta V\tilde{Q}u.$$

By (17) of Section 10,

$$\begin{aligned}|||\Gamma_\delta u|||_{\delta,\lambda,s} \leqslant C\, &|||\tilde{\Gamma}_\delta f_0|||_{\delta,\lambda,s-2} \\ &+ C\, |||\tilde{\Gamma}_\delta f_\nu|||_{\delta,\lambda,s-1} + C\,\delta^{-\eta}\, |||\tilde{\Gamma}_\delta \tilde{Q}u|||_{\delta,\lambda,s-1} \\ &+ C\,\delta^p \left(\|u\|_{-s_0} + \sum_{\nu \geqslant 0} \|f_\nu\|_{-s_0} \right).\end{aligned}$$

But

$$\begin{aligned}\delta^{-\eta}\, &|||\Gamma_\delta \tilde{Q}u|||_{\delta,\lambda,s-1} \\ &\leqslant C\,\delta^{-\eta}(|||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1} + \delta^p\, \|u\|_{-s_0})\end{aligned}$$

and

$$\delta^{-\eta}\, |||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1} \leqslant C\, |||\tilde{\Gamma}_\delta u|||_{\delta,\lambda,s-1/2}.$$

Thus, using the usual induction, we recover (17) of Section 9 for

$$(\tilde{A} - \lambda)u = f_0 + \sum_{\nu \geqslant 1} Z_\nu f_\nu.$$

Similarly, we recover (48) for the above equation, (49) for

$$(\tilde{A} - \lambda)u = f_0,$$

and (47) for

$$(\tilde{A} - \lambda)u = (S - \lambda)u.$$

Thus the main Lemma follows. ∎

Let us note, as a consequence of (17), that if $(\tilde{A} - \lambda)u = Yf$ (Y smooth subunit) then

$$\|\Gamma_\delta u\|_{L^\infty} \leqslant C\,\delta^{1/m}\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + \delta^p(\|f\|_{-s_0} + \|u\|_{-s_0})$$

for some fixed m (such that $\gamma \leqslant \delta^{1/m}$ for all x).

Global Main Lemma for $\tilde{A}$. *The Global Main Lemma holds with A replaced by $\tilde{A}$.*

Proof. Using $((K+A), u, u) \sim ((K+\tilde{A})u, u)$ the proof (a)–(f) goes through as before.

In view of the preceding remark, we also have

$$\|\Gamma_\delta(\tilde{A} - \lambda)^{-1}\, Yf\|_{L^\infty} \leqslant C\,\delta^{1/m}\, \|\tilde{\Gamma}_\delta f\|_{L^\infty} + C\,\delta^p\, \|f\|_{-s_0}.$$

Now the last line of (g) also goes through

$$\begin{aligned}(\tilde{A}-\lambda)^{-1} YY' &= (\tilde{A}-\lambda)^{-1}(L+B)(L+B)^{-1} YY' \\ &= (\tilde{A}+\lambda)^{-1}(\tilde{A}-\lambda-S+\lambda-Y\tilde{Q}+B)(L+B)^{-1} YY' \\ &= [I-(\tilde{A}-\lambda)^{-1}(S-\lambda)-(\tilde{A}-\lambda)^{-1} Y\tilde{Q} \\ &\quad + B(\tilde{A}+\lambda)^{-1}](L+B)^{-1} YY'. \quad \blacksquare\end{aligned}$$

The rest of the proof of Theorem 1.1 now goes through with A replaced by $\tilde{A}$.

PART II

12. Preliminaries

In what follows, $\mathscr{M}$ is a $2n-1$ $(n\geqslant 3)$ dimensional compact, pseudoconvex CR manifold. We assume $\bar{\partial}_b$ has closed range, and that in a neighborhood of a point $p_0 \in \mathscr{M}$ the Levi form is smoothly diagonalizable. Thus it is possible to choose $L_j \in T^{1,0}$ $(j=1, ..., n-1)$ smooth vector fields and T a real vector field such that L_j, $\bar{L}_k$, T span the tangent space of $b\Omega$ near p_0 and

$$[L_j, \bar{L}_k] = -i\,\delta_{jk}\lambda_j T + \sum a^l_{jk} L_l + \sum b^l_{jk} \bar{L}_l.$$

Pseudoconvex means that all $\lambda_j \geqslant 0$.

We also make the following "type m" assumption: for each $1 \leqslant i \leqslant n-1$, L_i, $\bar{L}_i$ and their commutators of length at most m, as well as all L_j, $\bar{L}_j$ $(i \neq j)$ span the tangent space of $\mathscr{M}$ near p_0. Under these hypotheses it follows from [K2] that $\Box_b$ satisfies a subelliptic estimate on $(0, q)$ forms for any $1 \leqslant q \leqslant n-2$.

Let ω_i be $(1, 0)$ forms dual to L_i and let us define the metric with respect to which the ω_i are orthonormal.

Our goal is to prove estimates for $\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}$ on $(0, q)$ forms $(1 \leqslant q \leqslant n-2)$.

If we represent a $(0, q)$ form as

$$\varphi = \sum f_I \overline{\omega^I},$$

where the sum is taken over all ordered q-tuples in $\{1, ..., n-1\}^q$ then

$$\bar{\partial}_b \varphi = \sum \varepsilon^{iI}_J \bar{L}_i f_I \overline{\omega^J} + O(1)\varphi$$

$$\bar{\partial}^*_b \varphi = -\sum \varepsilon^{iJ}_I L_i f_I \overline{\omega^J} + O(1)\varphi.$$

Here ε denotes the sign of a permutation and $O(1)$ is a matrix of smooth functions.

One computes that

$$\bar{\partial}^*_b \bar{\partial}_b \varphi = -\sum \varepsilon^{jJ}_{iI} (L_j \bar{L}_i f_I)\overline{\omega^J} + O(L, \bar{L}, 1)\varphi$$

and

$$\bar{\partial}_b \bar{\partial}^*_b \varphi = -\sum_{i \in I} (\bar{L}_i L_i f_I)\bar{\omega}^I + \sum_{i \neq j} \varepsilon^{iI}_{jJ} (\bar{L}_i L_j f_I)\bar{\omega}^J + O(L, \bar{L}, 1)\varphi,$$

where $O(L, \bar{L}, 1)$ denotes a matrix whose entries are linear combinations of the L's, $\bar{L}$'s and functions.

Finally, $\square_b = \square_d - \mathscr{V}$ where $\square_d$ is a diagonal operator,

$$\square_d\left(\sum f_I \bar{\omega}^I\right) = \sum (\square_I f_I)\bar{\omega}^I,$$

where

$$\square_I = \sum_{i \in I} L_i L^*_i + \sum_{j \notin I} \bar{L}^*_j \bar{L}_j$$

and $\mathscr{V} = O(L, \bar{L}, 1)$.

Our goal is to obtain estimates for $\square_I^{-1}$. Then estimates for $\square_b^{-1}$ will follow by iteration.

Let us choose coordinates $x_0, ..., x_{2n-2}$ near $p \in \mathscr{M}$ such that

$$L_1|_p = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\Bigg|_0$$

$$\vdots$$

$$L_{n-1}|_p = \left(\frac{\partial}{\partial x_{2n-3}} - i\frac{\partial}{\partial x_{2n-2}}\right)\Bigg|_0$$

$$T = \frac{\partial}{\partial x_0} \qquad \text{near} \quad p.$$

Let ξ_i be the dual coordinates. (Our Fourier transform is normalized so that the symbol of $(1/i)(\partial/\partial x_0)$ is ξ_0.)

Denote $\xi = (\xi_0, \xi')$. We proceed as in [K2, FK1] and consider the cones

$$\mathscr{C}^0 = \{\xi \in \mathbb{R}^{2n-1} \mid \text{either } |\xi'| > \tfrac{1}{2}|\xi_0| \text{ or } |\xi| \leqslant 2\}$$
$$\mathscr{C}^+ = \{\xi \in \mathbb{R}^{2n-1} \mid |\xi'| < |\xi_0| \text{ and } \xi_0 > 1\}$$
$$\mathscr{C}^- = \{\xi \in \mathbb{R}^{2n-1} \mid |\xi'| < |\xi_0| \text{ and } \xi_0 < -1\}.$$

Let U be a small neighborhood of p on $b\Omega$ and denote P_+, P_0, P_- pseudodifferential operators of order zero supported respectively in $U \times \mathscr{C}_+$, $U \times \mathscr{C}_0$, $U \times \mathscr{C}_-$.

Consider the following sets of derivatives, suitable for $\mathscr{C}_-$ or $\mathscr{C}_+$, acting on the left or right of $(\square_I)^{-1}$.

$$\mathscr{D}^I_{-l} = \{\bar{L}_i^*, \bar{L}_j^*, \bar{L}_j \mid i \in I, j \notin I\}$$
$$\mathscr{D}^I_{-r} = \{\bar{L}_i, \bar{L}_j^*, \bar{L}_j \mid i \in I, j \notin I\}$$
$$\mathscr{D}^I_{+l} = \{\bar{L}_j, \bar{L}_i^*, \bar{L}_i \mid i \in I, j \notin I\}$$
$$\mathscr{D}^I_{+r} = \{\bar{L}_j^*, \bar{L}_i^*, \bar{L}_i \mid i \in I, j \notin I\}.$$

THEOREM 12.1. *Let* $\varepsilon > 0$, $s \in \mathbb{R}$, $I \subset \{1, ..., n-1\}^q$. *Let* $D^I_{\pm l} \in \operatorname{span}(\mathscr{D}^I_{\pm l} \cup \{1\})$ $D^I_{\pm r} \in \operatorname{span}(\mathscr{D}^I_{\pm r} \cup \{1\})$.

Denote $\square_I^{-1}$ *a parametrix of* $\square_I$. *Then for any* $\varepsilon > 0$

(i) $\square_I^{-1} P_\pm$ *maps* $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s + 2/m - \varepsilon, p_0)$,

(ii) $D^I_{\pm l} \square_I^{-1} P_\pm$ *and* $\square_I^{-1} D^I_{\pm r} P_\pm$ *map* $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s + 1/m - \varepsilon, p_0)$, *and*

(iii) $D^I_{\pm l} \square_I^{-1} D^I_{\pm r} P_\pm$ *map* $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s - \varepsilon, p_0)$.

The proof will be given later. Notice that if $\tilde{P}_\pm$ are of the same type as $P_\pm$, only have symbols 1 on the support of the symbols of $P_\pm$ then $\tilde{P}_\pm \square_I^{-1} P_\pm$ differs from $\square_I^{-1} P_\pm$ by a smoothing operator of infinite order, by the arguments of [KN]. Also, notice that all the above hold trivially for $\square_I^{-1} P_0$, since $\square_I$ is elliptic in $U \times \mathscr{C}_0$.

COROLLARY 12.2 (Main Theorem). *Let* $\varepsilon > 0$, $s \in \mathbb{R}$. *Then* $\square_b^{-1}$ *(on* $(0, q)$ *forms,* $1 \leqslant q \leqslant n-2$*) maps* $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s + 2/m - \varepsilon, p_0)$, $\bar{\partial}_b \square_b^{-1}$, $\bar{\partial}_b^* \square_b^{-1}$, $\square_b^{-1} \bar{\partial}_b$, $\square_b^{-1} \bar{\partial}_b^*$ *map* $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s + 1/m - \varepsilon, p_0)$, *and* $\bar{\partial}_b^* \square_b^{-1} \bar{\partial}_b$, $\bar{\partial}_b \square_b^{-1} \bar{\partial}_b^*$ *map* $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s - \varepsilon, p_0)$. *In particular, the Szego kernel*

$$S = I - \bar{\partial}_b^* \square_b^{-1} \bar{\partial}_b$$

maps $\mathrm{Lip}(s, p_0)$ *into* $\mathrm{Lip}(s - \varepsilon, p_0)$.

Proof. For any $Z = \bar{L}_j^*$ or $\bar{L}_k$ and any I it is true that $Z \in \mathscr{D}^I_{+l}$ or

$Z \in \mathscr{D}^I_{+r}$ (or both), and that $Z \in \mathscr{D}^I_{-l}$ or $Z \in \mathscr{D}^I_{-r}$ (or both). It follows that any

$$\mathscr{K} = (\Box_{I_1})^{-1} Z_1 (\Box_{I_2})^{-1} Z_2 \cdots Z_{I_N} (\Box_{I_N})^{-1}$$

maps $\mathrm{Lip}(s, p_0)$ into $\mathrm{Lip}(s + (N+2)/m - \varepsilon, p_0)$. $D^{I_1}_{\pm l} \mathscr{K} P_\pm$, $\mathscr{K} D^{I_N}_{\pm r} P_\pm$ maps $\mathrm{Lip}(s, p_0)$ into $\mathrm{Lip}(s + (N+1)/m - \varepsilon, p_0)$ and $D^{I_1}_{\pm l} \mathscr{K} D^{I_N}_{\pm r} P_\pm$ maps $\mathrm{Lip}(s, p_0)$ into $\mathrm{Lip}(s + N/m - \varepsilon, p_0)$ for any fixed $\varepsilon > 0$. Thus by taking N sufficiently large

$$\Box_b^{-1} = (\Box_d - \mathscr{V})^{-1}$$

differs from

$$\Box_d^{-1} + \Box_d^{-1} \mathscr{V} \Box_d^{-1} + \cdots + \Box_d^{-1} (\mathscr{V} \Box_d^{-1})^N$$

by an operator which is regularizing of high order.

Notice that all derivatives in

$$\bar{\partial}_b \Box_d^{-1}, \quad \bar{\partial}_b^* \Box_d^{-1}, \quad \Box_d^{-1} \bar{\partial}_b$$

and $\Box_d^{-1} \bar{\partial}_b^*$ are suitable for both $\mathscr{C}_+$ and $\mathscr{C}_-$, in the sense of Theorem 12.1. By our previous observation, we are done. ▮

13. Some Lemmas

Let $\phi \neq I \subset \{1, ..., n-1\}$. Let $\eta = (1/100m)$ (m = type of $\mathscr{M}$). Define

$$\Sigma_{I,\delta} = \sum_{i \in I} (X_i^2 + Y_i^2) + \sum_{j \notin I} \left(\sum_{i \in I} \lambda_i + \delta^\eta \, |a^j_{ii}|^2 \right) (X_j^2 + Y_j^2)$$

(For $I = \{1, ..., k\}$ call $\Sigma_{I,\delta} = \Sigma_k$.)

Lemma 13.1. *For each $\Sigma_{I,\delta}$,*

$$\gamma(p, \delta) \leqslant C\, \delta^{1/m}$$

$$\lambda_i(p) \leqslant C \frac{\delta}{\gamma^2}.$$

Proof. Since $C(u - \Sigma_{k+1} u, u) \geqslant (u - \Sigma_k u, u)$ it suffices to prove the lemma for Σ_1. We will use the Fefferman–Phong method for computing a ball, which asserts that the ball of radius γ centered at (x_0, t_0) for

$$\frac{\partial^2}{\partial t^2} + \sum c_{ij}(x, t) \frac{\partial^2}{\partial x_i \, \partial x_j}$$

is

$$\{|t-t_0|<\gamma\}\times B(x_0,\gamma),$$

where $B(x_0,\gamma)$ is the ball associated to

$$\frac{1}{2\gamma}\int_{t_0-\gamma}^{t_0+\gamma}\sum c_{ij}(x,t)\frac{\partial^2}{\partial x_i\,\partial x_j}\,dt.$$

By the finite type assumption, there exists a monomial $M(X_1, Y_1)$ of degree $\leqslant m-2$ such that $M(\lambda_1)\neq 0$ near $p\in\mathcal{M}$. If we choose $\bar{X}_1 = X_1 + fY_1$, we can assume $(\bar{X}_1)^k\lambda_1\neq 0$ near p for some $k\leqslant m-2$. Define $\bar{X}_j = X_j + f_j\bar{X}_1$, $j\geqslant 2$, and $\bar{Y}_j = Y_j + g_j\bar{X}_1$ such that there exist coordinates in which $p=0$

$$\bar{X}_1=\frac{\partial}{\partial x_1}$$

$$\bar{X}_j|_0=\frac{\partial}{\partial x_j},\ \bar{Y}_j|_0=\frac{\partial}{\partial y_j}$$

and $\bar{X}_j$ $(j\geqslant 2)$, $\bar{Y}_j$ $(j\geqslant 1)$ have no $\partial/\partial x_1$ component. We may also assume

$$\bar{T}=\frac{\partial}{\partial t}.$$

The above changes do not affect the size of a ball of radius γ, so from now on we drop the bars.

By Fefferman and Phong we can assume all vector fields involved have polynomial coefficients, and the ball centered at 0 of radius γ is equivalent to

$$\{|x_1|<\gamma\}\times B(0,\gamma),$$

where B is associated to

$$\frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\left(Y_1^2+\sum_{j=2}^{n-1}(\lambda_1+\delta^\eta\,|a_{11}^j|^2)(X_j^2+Y_j^2)\right).$$

Using the fact that for polynomials $P(x_1)$ of bounded degree

$$\operatorname*{Av}_{|x_1|<\gamma}|P|^2\qquad\text{and}\qquad P^2(0)+\gamma^2\operatorname*{Av}_{|x_1|<\gamma}|P'|^2$$

are equivalent, $B(0, \gamma)$ is equivalent to the ball associated to

$$(Y_1|_{x_1=0})^2 + \gamma^2 \operatorname{Av}[X_1, Y_1]^2 + \operatorname{Av} \sum_{j=2}^{n-1} (\lambda_1 + \delta^\eta |a_{11}^j|^2)(X_j^2 + Y_j^2). \tag{1}$$

Now,

$$[X_1, Y_1] = -\frac{1}{2}\lambda_1 T + \sum_{j=2}^{n-1} O(|a_{11}^j|)X_j + \sum_{j=2}^{n-1} O(|a_{11}^j|)Y_j + O(Y_1).$$

Since $\gamma \leqslant c\,\delta^{1/2m}$ is clear (by inspecting the operator $X_1^2 + Y_1^2 + \sum_{k=2}^{n-1} (\lambda_1 X_k)^2$) we have $\gamma \ll \delta^{\eta/2}$ thus we obtain a ball equivalent to the one of (1) if we replace

$$\gamma^2 \operatorname{Av}[X_1, Y_1]^2 + \operatorname{Av} \sum_{j=2}^{n-1} \delta^\eta |a_{11}^j| (X_j^2 + Y_j^2)$$

by

$$\gamma^2 \operatorname{Av}\left(\lambda_1 \frac{\partial}{\partial t}\right)^2 + \operatorname{Av} \sum_{j=2}^{n-1} \delta^\eta |a_{11}^j|^2 (X_j^2 + Y_j^2).$$

In particular, $\gamma \operatorname{Av} \lambda_1(\partial/\partial t)$ is a subunit vector. If we expand in x_1,

$$X_j = X_j|_0 + O(\gamma)[X_1, X_j] + O(\gamma^2)[X_1, [X_1, X_j]] + \cdots.$$

We notice that $[X_1, X_j]$ has no T component, and the coefficients of T appear as $\gamma^2\lambda_1$, $\gamma^3\lambda_1'$, etc., thus are $O(\gamma^2 \operatorname{Av} \lambda_1)$. Thus the ball of (1) is equivalent to the ball of

$$D = (Y_1|_{x_1=0})^2 + \gamma^2 \operatorname*{Av}_{|x_1|<\gamma} \lambda_1^2 \frac{\partial^2}{\partial t^2} + \operatorname*{Av}_{|x_1|<\gamma} \sum_{j=2}^{n-1} (\lambda_1 + \delta^\eta |a_{11}^j|^2)((X_j|_{x_1=0})^2 + (Y_j|_{x_1=0})^2).$$

Since $(\partial/\partial x_1)^k \lambda_1(0) \neq 0$ for some $k \leqslant m-2$, $\operatorname{Av} \lambda_1 \geqslant c\gamma^{m-2}$. Thus D is elliptic. In fact

$$\|u\|^2 - (Du, u) \geqslant -c\gamma^2(\operatorname{Av} \lambda_1)^2 (\Delta u, u) \geqslant -c\gamma^{2m-2}(\Delta u, u).$$

We conclude that

$$\delta \geqslant c\gamma^2 \operatorname{Av} \lambda_1 \geqslant c'\gamma^2 \lambda_1(0)$$

and

$$\delta \geqslant c\gamma^m. \quad \blacksquare$$

LEMMA 13.2. *For every $N \geqslant 1$ there exists C such that for every $\delta > 0$ and $i \neq j$*

$$|a^j_{ii}|^2 \lambda_j \leqslant C(\lambda_i \delta^{-1} + \delta^N). \tag{2}$$

Proof. See also Proposition 2 in [DF]. Inspect the coefficient of $-\sqrt{-1}\, T$ in

$$[[L_i, \bar{L}_i], \bar{L}_j] + [[\bar{L}_j, L_i], \bar{L}_i] + [[\bar{L}_i, \bar{L}_j], L_i] = 0$$

to conclude that for some $f \in C^\infty$

$$\bar{L}_j(\lambda_i) + f\lambda_i = \lambda_j a^j_{ii}.$$

By taking real and imaginary parts it suffices to prove that if λ_i and $\lambda_j \geqslant 0$, f and a^j_{ii} are real, in $C^\infty(\mathbb{R})$ and

$$\frac{d\lambda_i}{dx} + f\lambda_i = a^j_{ii}\lambda_j$$

then (2) follows with bounds which depend only on the C^k (k large) norms of the functions involved. By replacing λ_i by $e^{\int f}\lambda_i$ we might as well assume

$$\frac{d\lambda_i}{dx} = a^j_{ii}\lambda_j. \tag{3}$$

Let us prove (2) at an arbitrary point, say 0.

If $a^j_{ii}(0) = 0$, there is nothing to prove. If $a^j_{ii}(0) \neq 0$, assume, without loss of generality, $a^j_{ii}(0) > 0$.

If $a^j_{ii}(x)$ is 0 at some x in $[-\delta, 0]$, let $\bar{x}$ be the largest such point. If not, let $-\bar{x} = \delta$. In either case, $\lambda_i' \geqslant 0$ in $[\bar{x}, 0]$, so

$$\begin{aligned}\frac{d\lambda_i}{dx}(0) &\leqslant C\left(|x_0|^{-1} \max_{[x_0, 0]} \lambda_i + \delta^N\right)\\ &\leqslant C\,(|x_0|^{-1}\lambda_i(0) + \delta^N)\end{aligned}$$

as one sees by taking an Nth order Taylor expansion of λ_i and using the fact that on a finite dimensional vector space all norms are equivalent.

If $a^j_{ii}(x_0)=0$, then, using (3)

$$\lambda_j(a^j_{ii})^2\,(0)=a^j_{ii}(0)\,\lambda'_i(0)$$
$$\leqslant C(\lambda_i(0)+\delta^N)$$

while if $x_0=-\delta$

$$\lambda_j(a^j_{ii})^2\,(0)\leqslant C\left(\frac{\lambda_i(0)}{\delta}+\delta^N\right). \quad ∎$$

14. Some Properties of Heat Kernels

We are on the compact manifold $\mathscr{M}$ and study a self-adjoint (pseudo) differential operator E which satisfies a subelliptic estimate

$$\|u\|^2+(Eu,u)\geqslant c\,\|u\|_\varepsilon.$$

E agrees locally with one of the following: $\tilde{A}$, $\square_I$, $\hat{\square}=\square_d+\mathscr{V}$ (where $\square_d$ is a diagonal matrix with entries $\square_{I_1},\ldots,\square_{I_k}$ and $\mathscr{V}$ has entries linear combinations of $\bar{L}_i$, $\bar{L}_i^*$ and functions). Say E is elliptic away from a small open set.

By comparing with a small power of the Laplacian, E has eigenvalues

$$\lambda_1\leqslant\lambda_2\leqslant\cdots$$
$$\lambda_j\geqslant cj^\eta$$

for some fixed $\eta>0$. Let ϕ_j be the corresponding eigenfunctions. Thus ϕ_j form an orthonormal basis of L^2.

Recall $\|\phi_j\|_{C^k}\leqslant C_k\lambda_j^{C'_k}$ by subellipticity.

Let us define, for $t>0$, e^{-tE} by the kernel

$$H(x,y,t)=\sum_{j=1}^{\infty}\phi_j(x)\,\bar{\phi}_j(y)e^{-t\lambda_j}.$$

Then

$$\left(\frac{\partial}{\partial t}-E\right)H(x,y,t)=0$$

$$\left\|\int H(x,y,t)\,f(y)\,dy\right\|_{L^2}\leqslant\|f\|_{L^2}$$

and

$$\int H(x,y,t)\,f(y)\,dy\rightarrow f(x)$$

in L^2, say. The above implies that for any $f \in C_0^\infty(b\Omega \times \mathbb{R})$,

$$\left(\frac{\partial}{\partial t} - E\right) \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int H(x, y, t')\, f(y, t-t')\, dt'\, dy$$

$$= \lim_{\varepsilon \to 0} \int H(x, y, \varepsilon)\, f(y, t-\varepsilon)\, dy$$

$$= f(x)$$

as distributions.

Thus $H(x, y, t-t')$ is the fundamental solution of $\partial/\partial t - E$. By arguments similar to Kohn–Nirenberg, $\partial/\partial t - E$ is hypoelliptic, thus $H(x, y, t)$ is C^∞ away from $x = y$, $t = 0$.

Also, if $f \in C_0^\infty(b\Omega)$

$$E \int_\varepsilon^1 \int H(x, y, t)\, f(y)\, dy\, dt$$

$$= \int_\varepsilon^1 \int \frac{\partial}{\partial t} H(x, y, t)\, dy\, dt$$

$$= \int H(x, y, 1)\, f(y)\, dy - \int H(x, y, \varepsilon)\, f(y)\, dy.$$

Thus $-\lim_{\varepsilon \to 0} \int_\varepsilon^1 H(x, y, t)\, dt$ is a parametrix for E. The semi-group property

$$\int H(x, z, t)\, H(z, y, t')\, dz = H(x, y, t+t')$$

is clear from the definition of H, and so is

$$H(x, y, t) = \frac{1}{2\pi i} \int_C e^{-t\lambda} \sum \frac{1}{\lambda_i - \lambda}\, \phi_i(x)\, \bar{\phi}_i(y)\, d\lambda$$

(as operators on L^2, say). Thus

$$e^{-tE} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (E - \lambda)^{-1}\, d\lambda.$$

We need one more property of e^{-tE}. Let $Q = \Gamma_\delta P_+$, $\tilde{Q} = \tilde{\Gamma}_\delta \tilde{P}_+$.

PROPOSITION 14.1.

$$\|Q e^{-tE} u\|_s \leqslant C(\|\tilde{Q} u\|_s + \delta^N \|u\|_{-N})$$

uniformly in t.

The proof follows from the following lemmas.

LEMMA 14.2. *For $\lambda \in C_B$, $B \gg 1$, as in Part* I,

$$\delta^N \|u\|_{-N} + \|\tilde{Q}(E-\lambda)u\|_s \geqslant C\,|\lambda|\,\|Qu\|_s.$$

We give the proof for $E = \Box_I$, the proof for $\hat{\Box}$ being similar and the proof for $E = \tilde{A}$ being similar to arguments in Part I.

We prove, by induction on s,

$$|\lambda|\,\|Qu\|_s + \sum \|Q\bar{L}_i u\|_{s+\varepsilon/2} \leqslant C\,\|\tilde{Q}(\Box_I - \lambda)u\|_s + C\,\delta^N\,\|u\|_{-N}.$$

Fix N. Start the induction with $s = -2N-1$, so that

$$\|\tilde{Q}\bar{L}_i u\|_s \leqslant C\,\delta^N\,\|u\|_{-N}.$$

Then we have

$$\begin{aligned} C'\,\|\tilde{Q}(\Box_I - \lambda)u\|_s^2 &+ C'\,\delta^N\,\|u\|_{-N} \\ &\geqslant C\,\|Q(\Box_I - \lambda)u\|_s^2 + C\,\delta^N\,\|u\|_{-N} \\ &= C\,\|Q\Box_I u\|_s^2 + C\,|\lambda|^2\,\|Qu\|_s^2 \\ &\quad -2C\,\mathrm{Re}\,\lambda(\Lambda^s Q\Box_I u, \Lambda^s Qu) \\ &\quad + C\,\delta^N\,\|u\|_{-N} \equiv S. \end{aligned}$$

Now, an easy integration by parts shows

$$\begin{aligned} (\Lambda^s Q\Box_I u, \Lambda^s Qu) &= (\Box_I \Lambda^s Qu, \Lambda^s Qu) \\ &\quad + O(\|\Lambda^s \tilde{Q}\bar{L}_i u\|\,\|\Lambda^s Qu\| + \|\Lambda^s Qu\|^2) \\ &= (\text{positive}) + O\left(\left(\frac{\text{l.c.}}{\lambda}\right)\|\Lambda^s \tilde{Q}\bar{L}_i u\|^2\right) \\ &\quad + \text{s.c.}\,|\lambda|\,\|\Lambda^s Qu\|^2. \end{aligned}$$

Thus for $|\mathrm{Im}\,\lambda| \geqslant c\,|\lambda|$, or $\mathrm{Re}\,\lambda = -B$

$$S \geqslant c\,\|Q\Box_I u\|_s^2 + c\,|\lambda|^2\,\|Qu\|_s^2.$$

Moreover,

$$\|Q\bar{L}_i u\|_{s+\varepsilon/2} \leqslant C\,\|\tilde{Q}\Box_I u\|_s + \|\tilde{Q}u\|_s + C\,\delta^N\,\|u\|_{-N}$$

is a well-known estimate from [K1]. Thus the case $s = -2N-1$ is proved. Now, suppose the case $s-\varepsilon/2$ is known. Then, arguing as before,

$$\begin{aligned} &C\,\|Q(\Box_I-\lambda)u\|_s^2 + C\,\delta^N\,\|u\|_{-N} \\ &\quad\geqslant \|Q\Box_I u\|_s^2 + |\lambda|^2\,\|Qu\|^2 - O(|\lambda|\,\|\tilde{Q}\bar{L}_i u\|_s\,\|Qu\|_s) \\ &\quad\geqslant \|Q\Box_I u\|_s^2 + c\,|\lambda|^2\,\|Qu\|^2 - C\,\|\tilde{Q}\bar{L}_i u\|_s^2. \end{aligned}$$

However,

$$\begin{aligned} &\|\tilde{\tilde{Q}}(\Box_I-\lambda)u\|_s^2 + C\,\delta^N\,\|u\|_{-N}^2 \\ &\quad\geqslant \|\tilde{\tilde{Q}}(\Box_I-\lambda)u\|_{s-\varepsilon/2}^2 + C\,\delta^N\,\|u\|_{-N}^2 \\ &\quad\geqslant c\,\|\tilde{Q}\bar{L}_i u\|_s^2 \end{aligned}$$

by induction.

LEMMA 14.3. *For $\lambda \in C_B$, $B \gg 1$,*

$$C\,\|(\Box_I-\lambda)u\|_{-N} \geqslant \|u\|_{-N}.$$

Proof. Let P_+, P_0, P_- as usual, with $P_+ + P_- + P_0 = I$. By the arguments of the preceding lemma, we have

$$\|u\|_{-N} + \|\tilde{P}(\Box_I-\lambda)u\|_{-N} \geqslant c\,|\lambda|\,\|Pu\|_{-N}$$

for P, $\tilde{P}$ one of $(P_+, \tilde{P}_+)$, $(P_-, \tilde{P}_-)$, $(P_0, \tilde{P}_0)$. Thus

$$\|\tilde{P}(\Box_I-\lambda)u\|_{-N} \geqslant c\,\|Pu\|_{-N}.$$

Summing up, we are done. ∎

15. ESTIMATES FOR THE HEAT KERNEL

THEOREM 15.1. *Let $\varepsilon > 0$, $I \subset \{1, ..., n-1\}^q$, and*

$$\begin{aligned} D^I_{+l} &\in \operatorname{span}(\mathscr{D}^I_{+l} \cup \{1\}) \\ D^I_{+r} &\in \operatorname{span}(\mathscr{D}^I_{+r} \cup \{1\}) \end{aligned}$$

and denote $\Box = \Box_I$. Then

(i) $\|\Gamma_\delta P_+ e^{-t\Box} f\|_{L^\infty}$
$\leqslant C\,\delta^{-\varepsilon}\,|\ln t|\,(\|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + \delta^p\,\|f\|_{-s_0})$

(ii) $\|\Gamma_\delta P_+ D^I_{+l} e^{-t\Box} f\|_{L^\infty}$
$\leqslant C\delta^{-\varepsilon} \min(t^{-1/2}, \delta^{-100}\,|\ln t|) \cdot (\|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + \delta^p\,\|f\|_{-s_0})$

(iii) $\|\Gamma_\delta P_+ e^{-t\Box} D^I_{+r} f\|_{L^\infty}$
$\leqslant C\,\delta^{-\varepsilon} \min(t^{-1/2}, \delta^{-100} |\ln t|) \cdot (\|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + \delta^p \|f\|_{-s_0})$

(iv) $\|\Gamma_\delta P_+ D^I_{+l} e^{-t\Box} D^I_{+r} f\|_{L^\infty}$
$\leqslant C\,\delta^{-\varepsilon} \min(t^{-1}, \delta^{-200}(\ln t)^2)(\|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + C\,\delta^p \|f\|_{-s_0})$

(v) $\|\Gamma_\delta P_+ \Sigma_{I,\delta} e^{-t\Box} f\|_{L^\infty}$
$\leqslant C\,\delta^{-\varepsilon} \min(t^{-1}, \delta^{-200}(\ln t)^2)(\|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + C\,\delta^p \|f\|_{-s_0})$

(vi) $\|\Gamma_\delta P_+ e^{-t\Box} \Sigma_{I,\delta} f\|_{L^\infty}$
$\leqslant C\,\delta^{-\varepsilon} \min(t^{-1}, \delta^{-200}(\ln t)^2)(\|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + C\,\delta^p \|f\|_{-s_0})$.

A similar statement holds with $+$ replaced by $-$. We will give the proof after a few remarks. In what follows ε denotes a number which can be made arbitrarily small, but not always the same.

If $H_{1,2}(x, y, t) \in C_0^\infty(\mathcal{M} \times \mathcal{M} \times (0, \infty))$, define

$$H_1 \# H_2(t) = \int_0^t \int_{\mathcal{M}} H_1(x, z, t-t')\, H_2(z, y, t')\, dt'.$$

Lemma 15.2. *Suppose*

$$\|H_0(t) f\|_{L^\infty} \leqslant C\,|\ln t|\, \|f\|_{L^\infty}$$
$$\|H_p(t) f\|_{L^\infty} \leqslant C t^{-p}\, \|f\|_\infty \qquad \textit{for} \quad 0 < p < 1$$
$$\|H_1(t) f\| \leqslant C \min(t^{-1}, N\,|\ln t|^2)\, \|f\|_\infty.$$

Then for $0 \leqslant p,\ q \leqslant 1,\ N > 1,\ 0 < t < 1$

$$\|H_p \# H_q(t) f\|_{L^\infty} \leqslant C t^{-p-q+1}\, |\ln N|\, |\ln t|\, \|f\|_{L^\infty}.$$

Proof. To fix ideas, let us take $p = 0$, $q = 1$ (the other cases are even more trivial). We must show

$$\int_0^t |\ln t'| \min((t-t')^{-1}, N\,|\ln(t-t')|^2)\, dt' \leqslant C\,|\ln N)\, |\ln t|.$$

Assume first $N^{-20} < t/2$. Then

$$\int_0^{t/2} |\ln t'|\, \frac{1}{t-t'}\, dt' \leqslant \frac{2}{t} \int_0^{t/2} |\ln t'|\, dt' \leqslant C\,|\ln t|$$

$$\int_{t/2}^{t-N^{-20}} |\ln t'|\, \frac{1}{t-t'}\, dt' \leqslant C\,|\ln t|\, |\ln N|$$

and

$$\int_{t-N^{-20}}^{t} |\ln t'|\, N\,|\ln(t-t')|^2\, dt' \leqslant C\,|\ln t|\, N^{-10}.$$

On the other hand, if $N^{-20} > t/2$, then

$$\int_0^{t/2} |\ln t'| \frac{1}{t-t'}\, dt' \leqslant C\, |\ln t|$$

and

$$\int_{t/2}^{t} |\ln t'|\, N\, |\ln(t-t')|^2\, dt'$$
$$\leqslant CNt\, |\ln t|^3 \leqslant CN^{-10}. \quad \blacksquare$$

Next, suppose we know Theorem 15.1 for all $|I| = q$. Fix $|I_1| = q, \ldots, |I_k| = q$ and let $\square_d$ be a $k \times k$ diagonal matrix with entries $\square_{I_1}, \ldots, \square_{I_k}$. Denote by $\mathbf{D}_{+l}$ any matrix whose rows have entries $D^{I_1}_{+l}, \ldots, D^{I_k}_{+l}$, and by $\mathbf{D}_{+r}$ any matrix whose columns have entries $D^{I_1}_{+l}, \ldots, D^{I_k}_{+l}$. Theorem 15.1 has an obvious translation to this new setting.

Finally, let $\mathscr{V}$ be a $k \times k$ matrix whose entries are linear combinations of L_i, L_i^*, and functions, and assume $\square_d + \mathscr{V} \equiv \hat{\square}$ is self-adjoint (and subelliptic, since $\square_d$ is).

LEMMA 15.3. *Parts* (i)–(iv) *of Theorem* 15.1 *hold with* $\square$ *replaced by* $\hat{\square}$ *and* D^I *replaced by* $\mathbf{D}$.

Proof.

$$\left(\frac{\partial}{\partial t} - \square_d - \mathscr{V}\right)^{-1} = \sum_{k=0}^{N} \left(\frac{\partial}{\partial t} - \square_d\right)^{-1} \left[\mathscr{V}\left(\frac{\partial}{\partial t} - \square_d\right)^{-1}\right]^{+k}$$
$$+ \text{arbitrarily smooth error.}$$

Since $e^{-(t-t')\square}$ is a parametrix for $\partial/\partial t - \square$, we have

$$e^{-t\hat{\square}} = e^{-t\square_d} + e^{-t\square_d} \# \mathscr{V} e^{-t\square_d}$$
$$+ \cdots + e^{-t\square_d} \# (\mathscr{V} e^{-t\square_d})^N$$
$$+ \text{smooth error.}$$

We used the fact that $\#$ represents iteration for kernels defined on $(\mathscr{M} \times [0, \infty))$, and that $(\partial/\partial t - \square)^{-1}$ is a smoothing of positive order on Sobolev spaces.

Now, $\mathscr{V} = \mathbf{D}_{+l} + \mathbf{D}_{+r}$ for a suitable choice of $\mathbf{D}$'s, thus any term in the sum of $e^{-t\hat{\square}}$ is a composition $(\#)$ of terms of the form $e^{-t\square_d}$, $\mathbf{D}_{+l} e^{-t\square_d}$, $e^{-t\square_d}\mathbf{D}_{+r}$, and $\mathbf{D}_{+l} e^{-t\square_d}\mathbf{D}_{+r}$.

By the previous lemma, the claim follows.

Proof of Theorem 15.1. The proof is by induciton on $|I|$. For $|I| = n-1$

$$\square_I = \sum L_i \bar{L}_i^* = \frac{1}{2} \sum (\bar{L}_i \bar{L}_i^* + \bar{L}_i^* \bar{L}_i)$$
$$+ \frac{1}{i} \sum \lambda_i T + O(L_i, L_j^*, 1).$$

Recall that the symbol of $(1/i) T$ is positive in $\mathscr{C}_+$, thus

$$\square_I P_+ \sim A_0 P_+$$

for an A_0 of type A, and the theorem follows from Theorem 1.1.

Let us assume Theorem 15.1 true for $|I| = n-k-1$, $k \geqslant 0$, and let us prove it, say, for

$$I = \{1, ..., n-k-2\}.$$

Let us define

$$\bar{\partial}_I f = \sum_{m-k-1}^{n-1} (\bar{L}_i f) \bar{\omega}_i$$
$$\bar{\partial}_I^* \left(\sum_1^{n-1} \varphi_i \bar{\omega}_i \right) = \sum_{n-k-1}^{n-1} \bar{L}_i^* \varphi_i$$
$$\bar{\partial}_I \left(\sum_1^{n-1} \varphi_i \bar{\omega}_i \right) = \sum_{n-k-1 \leqslant i < j \leqslant n-1} (\bar{L}_i \varphi_j - \bar{L}_j \varphi_i) \bar{\omega}_i \wedge \bar{\omega}_j$$

and

$$\bar{\partial}_I^* \left(\sum_{1 \leqslant i < j \leqslant n-1} \psi_{ij} \bar{\omega}_i \wedge \bar{\omega}_j \right)$$
$$= \sum_{n-k-1 < i < j \leqslant n-1} -\bar{L}_j^* \psi_{ij} \bar{\omega}_i + \bar{L}_i^* \psi_{ij} \bar{\omega}_j.$$

Note the following:

$$\square_I = \sum_1^{n-k-2} L_i L_i^* + \bar{\partial}_I^* \bar{\partial}_I$$
$$\hat{\square} = \sum_1^{n-k-2} L_i L_i^* + \bar{\partial}_I^* \bar{\partial}_I + \bar{\partial}_I \bar{\partial}_I^*$$

as a map of the space of forms $\sum_{n-k-1}^{n-1} \varphi_i \bar{\omega}_i$ to itself, is given by a $k \times k$ matrix,

$$\hat{\square} = \square_d + \mathscr{V},$$

where $\Box_d$ is a diagonal matrix with entries $\Box_{I\cup\{n-k-1\}}, \ldots, \Box_{I\cup\{n-1\}}$ and $\mathscr{V}$ has entries which are linear combinations of $\bar{L}_i$, L_i^*, and functions. The calculation for $\bar{\partial}_I\bar{\partial}_I^* + \bar{\partial}_I^*\bar{\partial}_I$ is identical to computing $\Box_b$ on (0, 1) forms on a $2k+1$ dimensional CR manifold.

Finally, note $\bar{\partial}_I\bar{\partial}_I = \bar{\mathscr{L}}$ is a matrix whose entries are linear combinations of $\bar{L}_i$ derivatives.

Notice

$$\begin{aligned}
\bar{\partial}_I\left(\frac{\partial}{\partial t} - \Box_I\right) &= \bar{\partial}_I\left(\frac{\partial}{\partial t} - \sum_1^{n-k-2} L_i\bar{L}_i^* - \bar{\partial}_I^*\bar{\partial}_I\right)\\
&= \left(\frac{\partial}{\partial t} - \sum_1^{n-k-2} \bar{L}_i L_i^* - \bar{\partial}_I\bar{\partial}_I^*\right)\bar{\partial}_I - \left[\bar{\partial}_I, \sum_1^{n-k-2} \bar{L}_i\bar{L}_i^*\right]\\
&= \left(\frac{\partial}{\partial t} - \sum_1^{n-k-2} \bar{L}_i L_i^* - \bar{\partial}_I\bar{\partial}_I^* - \bar{\partial}_I^*\bar{\partial}_I\right)\bar{\partial}_I\\
&\quad - \left[\bar{\partial}_I, \sum_1^{n-k-2} L_i\bar{L}_i^*\right] + \bar{\partial}_I^*\bar{\partial}_I\bar{\partial}_I\\
&= \left(\frac{\partial}{\partial t} - \hat{\Box}\right)\bar{\partial}_I + E.
\end{aligned}$$

Now,

$$\left[\bar{\partial}_I, \sum_1^{n-k-2} \bar{L}_i\bar{L}_i^*\right] = \sum \mathscr{V}_\nu \mathscr{N}_\nu,$$

where $\mathscr{V}_\nu$ is a matrix with entries $\bar{L}_i$, $\bar{L}_j^*$, and functions, $i, j \in \{1, \ldots, n-1\}$ and $\mathscr{N}_\nu$ is a matrix whose entries are linear combinations of $\bar{L}_i$, $\bar{L}_i^*$, and functions, $i \in I = \{1, \ldots, n-k-2\}$ only.

Also, $\bar{\partial}_I^*\bar{\partial}_I\bar{\partial} = \bar{\partial}_I^*\bar{\mathscr{L}}$ where $\bar{\mathscr{L}}$ has only $\bar{L}_i$ entries. Thus

$$E = \sum \mathscr{V}_\nu\mathscr{N}_\nu + \bar{\partial}_I^*\bar{\mathscr{L}}$$

can be represented as

$$\sum \mathbf{D}_{+r} D^I_{+l}.$$

We have thus

$$\begin{aligned}
\bar{\partial}_I\left(\frac{\partial}{\partial t} - \Box_I\right)^{-1} &= \left(\frac{\partial}{\partial t} - \hat{\Box}\right)^{-1}\bar{\partial}_I\\
&\quad - \left(\frac{\partial}{\partial t} - \hat{\Box}\right)^{-1} E\left(\frac{\partial}{\partial t} - \Box_I\right)^{-1}\\
&\quad + \text{smoothing}
\end{aligned} \tag{1}$$

$$\begin{aligned}\bar\partial_I\bar\partial_I\left(\frac{\partial}{\partial t}-\Box_I\right)^{-1}&=\bar\partial_I^*\left(\frac{\partial}{\partial t}-\hat\Box\right)^{-1}\bar\partial_I\\&\quad-\bar\partial_I^*\left(\frac{\partial}{\partial t}-\hat\Box\right)^{-1}E\left(\frac{\partial}{\partial t}-\Box_I\right)^{-1}\\&\quad+\text{smoothing}\end{aligned}\tag{2}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\sum_1^{n-k-2}\bar L_i\bar L_i^*\right)\left(\frac{\partial}{\partial t}-\Box_I\right)^{-1}\\&\quad=I_{\mathcal{M}\times\mathbb{R}}+\bar\partial_I^*\left(\frac{\partial}{\partial t}-\hat\Box\right)^{-1}\bar\partial_I\\&\quad\quad-\bar\partial_I^*\left(\frac{\partial}{\partial t}-\Box\right)^{-1}E\left(\frac{\partial}{\partial t}-\Box_I\right)^{-1}\\&\quad\quad+\text{smoothing,}\end{aligned}\tag{3}$$

where "smoothing" is of arbitarily high order.

Fix $0<\eta\leqslant 1/100m$ and denote

$$\Lambda_j=\sum_{i=1}^{n-k-2}(\lambda_i+\delta^\eta\,|a_{ii}^j|^2)\qquad(j=n-k-1,\ldots,n-1)$$

and let Λ be diagonal with entries Λ_j.

We also have

$$\begin{aligned}\bar\partial_I^*\Lambda\,\bar\partial_I\left(\frac{\partial}{\partial t}-\Box_I\right)^{-1}&=\bar\partial_I^*\Lambda\left(\frac{\partial}{\partial t}-\hat\Box\right)^{-1}\bar\partial_I\\&\quad-\bar\partial_I^*\Lambda\left(\frac{\partial}{\partial t}-\hat\Box\right)^{-1}E\left(\frac{\partial}{\partial t}-\Box_I\right)^{-1}.\end{aligned}\tag{4}$$

Let $0<c<1$, $N\gg 1$ to be chosen later, and define

$$\begin{aligned}A_1&=\frac{1}{2}\sum_{i=1}^{n-k-2}(\bar L_i^*\bar L_i+\bar L_i\bar L_i^*)+c\,\bar\partial_I^*\Lambda\,\bar\partial_I\\&\quad+\tilde\Gamma_\delta\tilde P_+\left(\sum_{i=1}^{n-k-2}[\bar L_i,\bar L_i^*]+\delta^N\frac{1}{i}T\right)\\&\quad+Q_0,\end{aligned}$$

where Q_0 is of order zero, chosen to make A_1 self-adjoint. Notice A_1 is subelliptic, and is of type $\tilde{A}$, as in Section 11. Explicitly,

$$\begin{aligned}
A_1 = & \sum_{i=1}^{n-k-2} (X_i^* X_i + Y_i^* Y_i) \\
& + c \sum_{j=n-k-1}^{n-1} (X_j^* \Lambda_j X_j + Y_j^* \Lambda_j Y_j) \\
& + \tilde{\Gamma}_\delta \tilde{P}_+ \left(\sum_{i=1}^{n-k-2} \lambda_i - c \sum_{j=n-k-1}^{n-1} \lambda_j \Lambda_j + \delta^N \right) \frac{1}{i} T \\
& + \sum_{i=1}^{n-k-2} \sum_{j=n-k-1}^{n-1} (a_{ii}^j L_j - \overline{a_{ii}^j L_j}) \\
& + \text{subunit vectors} + \text{function} + Q_0.
\end{aligned}$$

The second to last line is $\delta^{-\eta}$ subunit and the symbol of the third to last one is $\geqslant 0$ by Lemma 13.2, applied to δ^η (if c and N are properly chosen). Moreover,

$$\begin{aligned}
\Gamma_\delta P_+ A_1 \sim \Gamma_\delta P_+ & \left(\sum_{i=1}^{n-k-1} \bar{L}_i \bar{L}_i^* + c\, \bar{\partial}_I^* \Lambda\, \bar{\partial}_I + \delta^N \frac{1}{i} T + Q_0 \right) \\
\equiv & \Gamma_\delta P_+ B_1.
\end{aligned}$$

By (3) and (4)

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - B_1\right)\left(\frac{\partial}{\partial t} - \Box_I\right)^{-1} \\
& = I_{b\Omega \times \mathbb{R}} + \bar{\partial}_I^* \left(\frac{\partial}{\partial t} - \hat{\Box}\right)^{-1} \bar{\partial}_I \\
& \quad - c\, \bar{\partial}_I^* \Lambda \left(\frac{\partial}{\partial t} - \hat{\Box}\right)^{-1} \bar{\partial}_I \\
& \quad + \sum_{\mathbf{D}, D} \mathbf{D}_{+l} \left(\frac{\partial}{\partial t} - \hat{\Box}\right)^{-1} \mathbf{D}_{+r} D'_{+l} \left(\frac{\partial}{\partial t} - \Box_I\right)^{-1} \\
& \quad + O(\delta^N) + \text{smoothing} \\
& \quad + Q_0 \left(\frac{\partial}{\partial t} - \Box_I\right)^{-1}.
\end{aligned}$$

Composing with $\Gamma_\delta P_+(\partial/\partial t - A_1)^{-1}$ on the left,

$$\begin{aligned}\Gamma_\delta P_+ e^{-t\square_I} = \Gamma_\delta P_+ \Bigg(& [e^{-tA_1} + e^{-tA_1} \# \bar\partial_I^* e^{-t\hat\square}\, \bar\partial_I \\ & - c e^{-tA_1} \# \bar\partial_I^* \Lambda e^{-t\hat\square}\, \bar\partial_I] \\ & + \Bigg[\sum e^{-tA_1} \# \mathbf{D}_{+l} e^{-t\hat\square} \mathbf{D}_{+r} \# D^I_{+l} e^{-t\square_I} \\ & + \Gamma_\delta P_+ e^{-tA_1} \# Q_0 e^{-t\square_I} + O(\delta^N) + \text{smoothing}\Bigg]\Bigg) \\ = {} & \Gamma_\delta P_+([\text{Main term}] + [\text{Error term}]). \qquad (5)\end{aligned}$$

That $\Gamma_\delta P_+[\text{Main term}]$ satisfies (i), the subunit deriatives in (ii), and (v), of Theorem 15.1 follows from Lemma 15.2 using the estimates on e^{-tA_1} and the induction hypothesis on $e^{-t\hat\square}$ and its derivatives. (Notice $\bar\partial_I^*$ is of the form $\mathbf{D}_{+l}$.)

For the $\bar L_j$ derivatives in (ii) and the error term, rewrite (1) as

$$\begin{aligned}\bar\partial_I e^{-t\square_I} = e^{-t\hat\square}\, \bar\partial_I - \sum e^{-t\hat\square}\, \mathbf{D}_{+r} \# D^I_{+l} e^{-t\square_I} \\ + \text{smoothing}. \qquad (6)\end{aligned}$$

Now, any D^I_{+l} is either a subunit vector for the differential part of A_1, in which case we use (5), or else a component of $\bar\partial_I$ (that is, $\bar L_j$ for $n-k-1 \leqslant j \leqslant n-1$), in which case we use (6). Parts (iii) and (vi) are proved by the adjoint of the above argument.

Any D^I_{+r} is the adjoint of a D^I_{+l}, and to estimate them use the adjoint of (5) and (6).

Feeding (5), (6), (5)*, (6)* into the error terms a finite number of times, we end up with error terms of the form

$$e^{-t\square_1} \# D_2 e^{-t\square_2} \# \cdots \# D_n e^{-t\square_n},$$

where D_i are $\bar L_i^*$ or $\bar L_i$. Such a term can be made arbitrarily smooth.

Thus we have proved Theorem 15.1, (i), (ii), (iii), (v), (vi).

The missing point, (iv), which apparently cannot be proved by the above argument, follows from the semi-group property $e^{-t\square_I} e^{-t\square_I} = e^{2t\square_I}$. ∎

16. Proof of Theorem 12.1

Taking

$$(\square_I)^{-1} = \int_0^1 e^{t\square_I}\, dt$$

we get from Theorem 15.1, (iv), (v), (vi)

$$\|\Gamma_\delta P_+ \Sigma_{I,\delta} \Box_I^{-1} f\|_{L^\infty} + \|\Gamma_\delta P_+ D^I_{+I} \Box_I^{-1} D^I_{+r} f\|_{L^\infty} + \|\Gamma_\delta P_+ \Box_I^{-1} \Sigma_{I,\delta} f\|_{L^\infty}$$
$$\leqslant C\delta^{-\varepsilon} \|\tilde{\Gamma}_\delta \tilde{P}_+ f\|_{L^\infty} + C\delta^N \|f\|_{-N}$$

which imply parts (i), (iii) of the theorem, as well as (ii) when the D^I's are subunit for $\Sigma_{I,\delta}$.

For $\bar\partial_I(\Box_I)^{-1}$ and $(\Box_I)^{-1}\bar\partial_I^*$ it seems necessary to redo the induction, using

$$\bar\partial_I(\Box_I)^{-1} = \hat\Box^{-1}\bar\partial_I + \hat\Box^{-1}\mathbf{D}_{+r} D^I_{+I} \Box_I^{-1}. \tag{1}$$

The estimate is clear for the main term, and the only unknown error terms are of the form $\bar\partial_I(\Box_I)^{-1}$, so (1) can be fed into the error terms, completing the proof.

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